Jacobian

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Transformation in 1D

$$\int_C f(x)dx = \int_{C*} f\left(g(t)\right) \cdot g'(t)dt$$

Transformation in 2D

$$\begin{cases} u = u\left(x,y\right) \\ v = v\left(x,y\right) \end{cases}$$

Theorem. *Let D and D^* be elementary regions in (respectively) the xy-plane and uv-plane. Suppose $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a coordinate transformation of class C^{-1} that map D on D^* (one-to-one). If $f : D \longrightarrow \mathbb{R}$ is an integrable function and we use transformation T to make the substitution $x = x(u,v)\; y = y(u,v)$ then*

$$\iint_D f(x,y)dxdy = \iint_{D*} f\left(x(u,v)\,,\; y(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv$$

Where the Jacobian of transform is the determinant of the derivative of the transform matrix T

$$\frac{\partial(x,y)}{\partial(u,v)} = det\,DT(u,v) = det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \quad \textit{It means : area } dxdy = Jdudv$$

An illustration

Consider $|d\overrightarrow{x} \times d\overrightarrow{y}|$

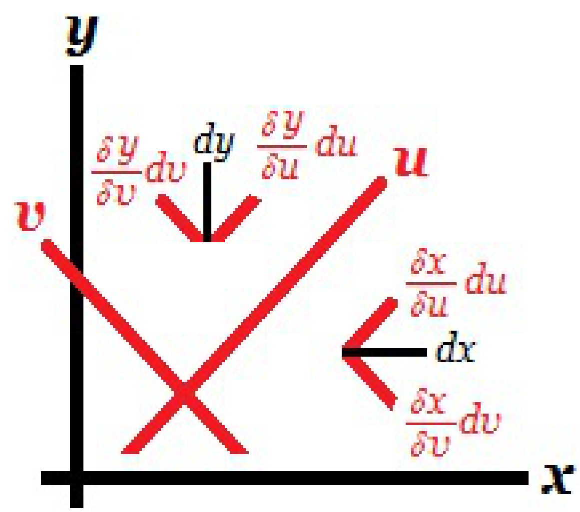
$$dA_{xy} = |d\overrightarrow{x} \times d\overrightarrow{y}| = \det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx & 0 & 0 \\ 0 & dy & 0 \end{bmatrix}\left|\hat{n}_{xy}\right| = dxdy$$

Consider dA_{uv}

$$dA_{uv} = |d\overrightarrow{u} \times d\overrightarrow{v}| = \det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial x}{\partial u}du & \frac{\partial x}{\partial v}dv & 0 \\ \frac{\partial y}{\partial u}du & \frac{\partial y}{\partial v}dv & 0 \end{bmatrix}\left|\hat{n}_{xy}\right|$$

$$= \left|\frac{\partial x}{\partial u}du\frac{\partial y}{\partial v}dv - \frac{\partial y}{\partial u}du\frac{\partial x}{\partial v}dv\right|$$

$$= \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right|dudv = \left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix}\right|dudv = \frac{\partial\left(x,y\right)}{\partial\left(u,v\right)}dudv$$



Remark. $dA_{xy} \neq dA_{uv}$, but they are in proportional, the proportional constant is the Jacobian

Property
$$\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(u,v)}=1 \qquad \iff \qquad \frac{\partial(u,v)}{\partial(x,y)}=\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^{-1}$$

Proof.
$$\begin{cases} x=x(u,v) \\ y=y(u,v)\end{cases} \qquad \begin{cases} u=u(x,y) \\ v=v(x,y)\end{cases}$$

Then
$$\begin{cases} dx=\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv \\ dy=\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv\end{cases} \qquad \begin{cases} du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy \\ dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy\end{cases}$$

Thus
$$\begin{cases} du=\frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv\right)+\frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv\right) \\ dv=\frac{\partial v}{\partial x}\left(\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv\right)+\frac{\partial v}{\partial y}\left(\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv\right)\end{cases}$$
$$\Rightarrow \begin{cases} du=\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial u}\right)du+\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial v}\right)dv \\ dv=\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial v}\right)dv+\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial u}\right)du\end{cases}$$

Since du and dv are independent , i.e. $du=1\cdot du+0\cdot dv \qquad dv=1\cdot dv+0\cdot du$

Thus $\frac{\partial u}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial u}=1 \quad \frac{\partial u}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial v}=0 \quad \frac{\partial v}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial v}=1 \quad \frac{\partial v}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial u}=0$

Hence, consider the original statement $\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(u,v)}$

$$\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)}=\det\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}\det\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}$$

Using a property of Determinant (Proof skiped)

$$\det\begin{bmatrix} a & b \\ c & d\end{bmatrix}\cdot\det\begin{bmatrix}\alpha & \beta \\ \gamma & \delta\end{bmatrix}=\det\begin{bmatrix} a\alpha+b\gamma & a\beta+b\delta \\ c\alpha+c\gamma & c\beta+d\delta\end{bmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(u,v)}=\det\begin{pmatrix}\frac{\partial u}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial v}\end{pmatrix}=\det I_2=1$$

□

General Jacobians $\quad dx_1dx_2...dx_n=\left|\frac{\partial(x_1,x_2,...,x_n)}{\partial(y_1,y_2,..,y_n)}\right|dy_1dy_2...dy_n$

i.e. The integral

$$\int\int...\int_R f(x_1,x_2,...,x_n)\,dx_1dx_2...dx_n=\int\int...\int_{R'} f(y_1,y_2,...,y_n)\left|\frac{\partial(x_1,x_2,...,x_n)}{\partial(y_1,y_2,..,y_n)}\right|dy_1dy_2...dy_n$$

An famous application of Jacobian $\int_{-\infty}^{\infty}e^{-Ax^2}dx$

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