SET THEORETIC DEFINITIONS OF SOME MATHEMATICAL OBJECTS

1. Group

Definition 1. Consider a set G with an *binary operation* $\circ$ (that operate on two element a and b in G to produce another element c). The structure $(G, \circ)$ is a *group* if it satisfy the $group\, axiom$

$\mathcal{G}1$. *Closure.* For any two elements in G , the result of the operation $\circ$ is also in G. In symbols : $\forall a, b \in G$, $c = a \circ b \in G$.

$\mathcal{G}2$. *Associativity.* For any three elements in G, the order of operations performed does not matter if the sequence of the operands is not changed. In symbols : $\forall a, b, c \in G$, $a \circ (b \circ c) = (a \circ b) \circ c$.

$\mathcal{G}3$. $Identity$. There exists an element e in G, for all element a in G, $e \circ a = a \circ e = a$. Such an element is called $identity element$.

$\mathcal{G}4$. $Inverse$. For all element a in G, there exists an element b in G such that $a \circ b = b \circ a = e$, b is called the $inverse$ of a, denoted as a^{-1}.

***:** Such definition does not involve $a \circ b = b \circ a$ (commutativity)

2. Ring

Definition 2. Consider a set R with two *binary operation* $+$, called *addition*, and $\times$, called multiplication. The structure $(R, +, \times)$ is a *ring* if it satisfy the $ring\, axiom$

$\mathcal{R}1$. $Closure\, on +$. For any two elements in R , the result of the operation $+$ is also in R. In symbols : $\forall a, b \in R$, $c = a + b \in R$.

$\mathcal{R}2$. $Associativity\, on +$. For any three elements in R, the order of operations performed does not matter if the sequence of the operands is not changed. In symbols : $\forall a, b, c \in R$, $a + (b + c) = (a + b) + c$.

$\mathcal{R}3$. $Commutativity\, on +$. For any two elements a, b in R, $a + b = b + a$

$\mathcal{R}4$. $Additive\, Identity$. There exists an element e_+ in R, for all element a in R, $e_+ + a = a + e_+ = a$. Such an element is called $additive\, identity element$. e_+ is usually denoted as 0.

$\mathcal{R}5$. $Additive\, Inverse$. For all element a in R, there exists an element b in R such that $a + b = b + a = 0$, b is called the $additive\, inverse$ of a, denoted as $-a$.

$\mathcal{R}6$. $Closure\, on \times$. For any two elements in R , the result of the operation $\times$ is also in R. In symbols : $\forall a, b \in R$, $c = a \times b \in R$.

$\mathcal{R}7$. $Associativity\, on \times$. For any three elements in R, the order of operations performed does not matter if the sequence of the operands is not changed. In symbols : $\forall a, b, c \in G$, $a \times (b \times c) = (a \times b) \times c$.

$\mathcal{R}8$. $Multiplicative\, Identity$. There exists an element $e_\times$ in R, for all element a in R, $e_\times \circ a = a \circ e_\times = a$. Such an element is called $multiplicative\, identity element$. $e_\times$ is usually denoted as 1.

$\mathcal{R}9$. $Left\, distributive$. For any three elements in R, $a \times (b + c) = (a \times b) + (a \times c)$.

$\mathcal{R}10$. $Right\, distributive$. For any three elements in R, $(b + c) \times a = (b \times a) + (c \times a)$.

***:** In simple words, $(R, +)$ is an *abelian group*.(Commutative group), $(R, \times)$ is a *monoid* (semigroup with identity), and multiplication distributes over addition

3. Ideal

Definition 3. Consider a *ring* $(R, +, \times)$, a subset I of R is a *ideal* if it satisfy the $ideal\, axiom$.

$\mathcal{I}1$. $Subgroup\, on +$. $(I, +)$ is a subgroup of $(R, +)$. (I is a subset of R)

$\mathcal{I}2$. $Left\, ideal$. For any elements in I, for any element in R , the left-multiplication of these two element is also in I : $\forall \alpha \in I$, $\forall a \in R$, $a \times \alpha \in I$.

$\mathcal{I}2$. $Right\, ideal$. For any elements in I, for any element in R , the right-multiplication of these two element is also in I : $\forall \alpha \in I$, $\forall a \in R$, $\alpha \times a \in I$.

***:** In simple words, I "absorbs the multiplication by elements of R ".

4. Field

Definition 4. Consider a set F with two *binary operation* $+$, called *addition*, and $\times$, called multiplication. The structure $(F,+,\times)$ is a $field$ if it satisfy the $field\,axiom.$

$\mathcal{F}1$. *Ring.* $(F,+,\times)$ is a $ring.$ $(\mathcal{R}1-\mathcal{R}10)$

$\mathcal{F}2$. $Commutativity\,on\times.$ For any two elements a,b in F, $a\times b=b\times a$

$\mathcal{F}3$. $Multiplicative\,Identity.$ There exists an element $e_{\times}$ in F, for all element a in F, $e_{\times}\circ a=a\circ e_{\times}=a$. Such an element is called $multiplicative\,identityelement.$ $e_{\times}$ is usually denoted as 1.

5. Topology

Definition 5. Consider a set X and a family of subset of T . The structure (X,T) is a *topology space* and T is *topology* if it satisfy the $topology\,axiom$

$\mathcal{T}1$. The set X and null set are in T : $X,\varnothing\in T$

$\mathcal{T}2$. For any two element in T, their union is also in T

$\mathcal{T}3$. For any finite elements in T, their intersection is also in T

- ***:** The members of T are called *open set* in X. A subset of X is *closed* if its complement is in T.
- ***:** A subset of X may be open, closed, both (clopen), or neither.
- ***:** X itself and $\varnothing$are always both closed and open.

6. Vector Space

Definition 6. Consider a field $(F,+,\times)$, a set X is a *vector space* over F if it satisfy the $vector\,space\,axiom$

$\mathcal{V}1$. $Closure\,on+.$ $\forall a,b\in X$, $a+b\in X$

$\mathcal{V}2$. $Associativity\,on+.$ $\forall a,b,c\in X$, $a+(b+c)=(a+b)+c.$

$\mathcal{V}3$. $Commutativity\,on+.$ $\forall a,b\in X$, $a+b=b+a$

$\mathcal{V}4$. $Additive\,Identity.$ There exists an element e_{+X} in X, $\forall a\in X$, $e_{+X}+a=a+e_{+X}=a$. Such an element is called $additive\,identityelement$, or $zero\,vector.$ e_{+X} is usually denoted as 0.

$\mathcal{V}5$. $Additive\,Inverse.$ $\forall a\in X$, there exists an element b in X such that $a+b=b+a=0$, b is called the $additive\,inverse$ of a, or *negative vecotr*, denoted as $-a$.

$\mathcal{V}6$. $Closure\,on\,scalar\,multiplication.$ $\forall a\in X$, and $\forall\alpha\in F$, $\alpha\times a\in X$.

$\mathcal{V}7$. $Scalar\,Multiplicative\,Identity.$ $\forall a\in X$, there exists an element $e_{\times F}$ in F, $e_{\times F}\circ a=a$. Such an element is called $scalar\,multiplicative\,identityelement.$ $e_{\times F}$ is usually denoted as 1.

$\mathcal{V}8$. $Associativity\,on\,scalar\,multiplication.$ $\forall a\in X$, $\forall\alpha,\beta\in F$, $\alpha\times(\beta\times a)=(\alpha\times\beta)\times a$.

$\mathcal{V}9$. $Left-Distributive\,I.$ $\forall a\in X$, $\forall\alpha,\beta\in F$, $(\alpha+\beta)\times a=(\alpha\times a)+(\beta\times a)$.

$\mathcal{V}9$. $Left-Distributive\,II.$ $\forall a,b\in X$, $\forall\alpha\in F$, $\alpha\times(a+b)=(\alpha\times a)+(\alpha\times b)$.

- ***:** The scalar multiplication $\alpha a=a\alpha$ is not in the definition. But since $(F,\times)$ is commutative, we can define $a\alpha\triangleq\alpha a$