Mix Derivative

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This is a short illustration of the following equality

$$\frac{\partial f(x,y)}{\partial x \partial y} = \frac{\partial f(x,y)}{\partial y \partial x}$$

Consider the LHS

$$\frac{\partial f(x,y)}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f(x,y)}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}\left[\lim_{\delta y \to 0} \frac{f\left(x, y+\delta y\right) - f\left(x,y\right)}{\delta y}\right]$$

$$= \lim_{\delta x \to 0} \frac{\left\{\lim_{\delta y \to 0} \frac{f\left(x+\delta x, y+\delta y\right) - f\left(x+\delta x, y\right)}{\delta y}\right\} - \left\{\lim_{\delta y \to 0} \frac{f\left(x, y+\delta y\right) - f\left(x,y\right)}{\delta y}\right\}}{\delta x}$$

$$= \lim_{\delta y \to 0} \frac{\lim_{\delta x \to 0} \frac{f\left(x+\delta x, y+\delta y\right) - f\left(x+\delta x, y\right)}{\delta x} - \lim_{\delta x \to 0} \frac{f\left(x, y+\delta y\right) - f\left(x,y\right)}{\delta x}}{\delta y}$$

$$= \lim_{\delta y \to 0} \frac{\lim_{\delta x \to 0} \frac{f\left(x+\delta x, y+\delta y\right) - f\left(x+\delta x, y\right) - f\left(x, y+\delta y\right) + f(x,y)}{\delta x}}{\delta y}$$

$$= \lim_{\delta y \to 0} \frac{\lim_{\delta x \to 0} \frac{f\left(x+\delta x, y+\delta y\right) - f\left(x, y+\delta y\right)}{\delta x} - \lim_{\delta x \to 0} \frac{f\left(x+\delta x, y\right) - f(x,y)}{\delta x}}{\delta y}$$

$$= \lim_{\delta y \to 0} \frac{\frac{\partial f\left(x, y+\delta y\right)}{\partial x} - \frac{\partial f\left(x,y\right)}{\partial x}}{\delta y}$$

$$= \frac{\partial}{\partial y}\frac{\partial f(x,y)}{\partial x}$$

$$= \frac{\partial f(x,y)}{\partial y \partial x}$$

$-END-$