Stirling's Approximation of $\ln N!$

March 21, 2013

For **large** N **:**

$$\ln N! \simeq N\ln N - N$$

Proof. First,

$$\ln N! = \ln N(N-1)...1 = \ln N + \ln(N-1) + ... + \ln 1 = \sum_{n=1}^{N} \ln n$$

Consider

$$\sum_{n=1}^{N} \ln n \Delta n = \sum_{n=1}^{N} N \ln n \frac{\Delta n}{N}$$

$$= \sum_{1}^{N} N \ln\left(\frac{nN}{N}\right) \Delta\frac{n}{N} = \sum_{1}^{N} N\left(\ln\frac{n}{N} + \ln N\right)\Delta\frac{n}{N} = \sum_{1}^{N}\left(N\ln\frac{n}{N} + N\ln N\right)\Delta\frac{n}{N}$$

Take the limit $N \to \infty$, $\Delta\frac{n}{N} = \mathrm{d}x$

$$= \int_{\frac{1}{N}}^{1} (N\ln x + N\ln N)\,\mathrm{d}x$$

$$= \int_{\frac{1}{N}}^{1} N\ln x\mathrm{d}x + N\ln N\int_{\frac{1}{N}}^{1}\mathrm{d}x$$

$$= N\left[x\ln x - x\right]_{\frac{1}{N}}^{1} + N\ln N\left[1 - \frac{1}{N}\right] == [-N + \ln N + 1] + [N\ln N - \ln N]$$

$$= N\ln N - N + 1$$

Hence, for **large** N

$$\ln N! \approx N\ln N - N$$

−END−

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