Taylor's Theorem

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Taylor's Series can be treated in 2 ways :

- An expansion form for a function in certain value
- An asymptotic series with increasing accuracy

Taylor's Theorem

For a infinitely differentiable function f in a neighborhood of a value a , the series expansion is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f"(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\eta)}{(n+1)!}(x-a)^{n+1}$$

Maclaurin Series

If $a = 0$, it is called Maclaurin series

$$f(x) = f(0) + f'(0)x + \frac{f"(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + ... + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\eta)}{(n+1)!}x^{n+1}$$

Laurent Series

If a is complex number, it is called Laurent series (useful in complex analysis)

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n(z-z_0)^n +}_{\text{Laurent series}} \sum_{n=-1}^{-\infty} b_n(z-z_0)^n$$

Incremental Form

If let $x - a = h$, so $x = h + a$, then the incremental form is

$$f(x) = f(a+h) = f(a) + f'(a)h + \frac{f"(a)}{2!}h^2 + \frac{f^{(3)}(a)}{3!}h^3 + ... + \frac{f^{(n)}(a)}{n!}h^n + \frac{f^{(n+1)}(\eta)}{(n+1)!}h^{n+1}$$

Since h is small, so for term having h^3 or above basically goes to zero, then the value $f(a)$ is thus

$$f(a+h) \approx f(a) + f'(a)h + \frac{f"(a)}{2!}h^2$$

If also considering $h^2 = 0$ (Linearize the Taylor's Series)

$$f(a+h) \approx f(a) + f'(a)h$$

Assume $f(x) = 0$ at $x = r$, and $r = a + h$

$$0 = f(r) = f(a+h) \approx f(a) + f'(a)h$$

thus

$$h \approx -\frac{f(a)}{f'(a)} \qquad f'(a) \neq 0$$

Then

$$a + h \approx a - \frac{f(a)}{f'(a)}$$

Generalize

$$x_{n+1} = x_n + h \approx x_n - \frac{f(x_n)}{f'(x_n)}$$

This is the Newton-Raphson's Method for solving nonlinear equations

Taylor's Series for function of 2 variable

$$\begin{aligned} f(x,y) = f(a,b) + (x-a)\frac{\partial f(a,b)}{\partial x} + (y-b)\frac{\partial f(a,b)}{\partial y} \\ + \frac{1}{2!}\left\{(x-a)^2\frac{\partial^2 f(a,b)}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f(a,b)}{\partial x \partial y} + (y-b)\frac{\partial^2 f(a,b)}{\partial y^2}\right\} + \ldots \end{aligned}$$

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