Tricks on Integration of e^{-Ax^2}

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The integration of $\exp(-Ax^2)$ will be found during the derivation of Gaussian Distribution.

Tricks on integration of exp$(-Ax^2)$

$$I = \int_0^\infty e^{-Ax^2} dx = \left\{ \int_0^\infty e^{-Ax^2} dx \right\}^{\frac{2}{2}} = \left\{ \int_0^\infty e^{-Ax^2} dx \cdot \int_0^\infty e^{-Ay^2} dy \right\}^{\frac{1}{2}} = \left\{ \int_0^\infty \int_0^\infty e^{-A(x^2+y^2)} dxdy \right\}^{\frac{1}{2}}$$

Coordinate Transform

Rectangular $(x\ ,\ y) \longrightarrow$ **Polar** $(r\ ,\ \theta)$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2 + y^2 = r^2 \\ \theta = \tan^{-1}\dfrac{y}{x} \end{cases}$$

$$dxdy = rdrd\theta \ \&\ x, y \in [0, \infty) \longrightarrow \begin{cases} r \in [0, \infty) \\ \theta \in [0, \dfrac{\pi}{2}] \quad \text{First Quardant} \end{cases}$$

Remark. The r comes from the *Jacobian* :

$$J(r, \theta) = \frac{\partial(x, y)}{\partial(r, \theta)} = \det \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \end{bmatrix} = \det \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} = r$$

Therefore

$$I = \left\{ \int_0^\infty \int_0^{\pi/2} e^{-Ar^2} rdrd\theta \right\}^{\frac{1}{2}}$$

$$rdr = \frac{1}{2}dr^2$$

$$I = \left\{ \frac{1}{2} \int_0^\infty \int_0^{\pi/2} e^{-Ar^2} dr^2 d\theta \right\}^{\frac{1}{2}} = \sqrt{\frac{1}{2} \int_0^{\pi/2} \left[\frac{e^{-Ar^2}}{-A} \right]_0^\infty d\theta} = \sqrt{\frac{1}{2A} \int_0^{\pi/2} d\theta} = \sqrt{\frac{\pi}{4A}}$$

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