Gamma Function and Gamma-Poisson Distribution

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1 The Gamma Function

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$$

Some illustrations

$\Gamma(1)$

$$\Gamma(1) = \int_0^\infty u^{1-1} e^{-u} du = \int_0^\infty e^{-u} du$$

Let $u = -v$, $du = -dv$, $u = 0 \to v = 0$, $u = +\infty \to v = -\infty$

$$\Gamma(1) = -\int_0^{-\infty} e^v dv = -\left[e^v\right]_0^{-\infty} = 1$$

$\Gamma(2)$

$$\Gamma(2) = \int_0^\infty u^1 e^{-u} du = -\int_0^\infty u de^{-u} = \underbrace{-ue^{-u}|_0^\infty}_{0} + \underbrace{\int_0^\infty e^{-u} du}_{\Gamma(1)} = \Gamma(1) = 1$$

Remark. $ue^{-u}|_0^\infty = \lim_{u \mapsto \infty} \frac{u}{e^u} - \frac{u}{e^u}|_{u=0} = \lim_{u \mapsto \infty} \frac{u}{e^u} = \lim_{u \mapsto \infty} \frac{1}{e^u} = 0 \qquad (L'Hospital)$

Property : $\Gamma(a+1) = a\Gamma(a)$

$$\Gamma(a+1) = \int_0^\infty u^a e^{-u} du = -\int_0^\infty u^a de^{-u}$$

$$\Longleftrightarrow \quad \underbrace{-\left[\frac{u^a}{e^u}\right]_0^\infty}_{0} + a\int_0^\infty u^{a-1} e^{-u} du = a\Gamma(a)$$

Remark. $\left[\frac{u^a}{e^u}\right]_0^\infty = \lim_{u \to \infty} \frac{u^a}{e^u} - \left(\frac{u^a}{e^u}\right)_{u=0} = \lim_{u \to \infty} \frac{u^a}{e^u} = \lim_{u \to \infty} \frac{au^{a-1}}{e^u} = \lim_{u \to \infty} \frac{a(a-1)u^{a-2}}{e^u} = ... = 0 \quad (L'Hospital)$

Generalize, for $n \in \mathbb{N}^+$ $\qquad \Gamma(n+1) = n\Gamma(n) = ... = n(n-1)(a-2)...\Gamma(1) = n!$

$\Gamma(0)$

$$\Gamma(0) = \int_0^{\infty} u^{-1}e^{-u}du = ?$$

The function decays very quickly for large u , and it has a decaying behaviour similar to $\dfrac{1}{x}$ for small x

Using inequality $\dfrac{1}{e^a} \geq \dfrac{1}{e}$ for $a \in [0,1]$

$$\lim_{\epsilon \mapsto 0} \int_{\epsilon}^{1} u^{-1}e^{-u}du \geq \lim_{\epsilon \mapsto 0} \frac{1}{e} \int_{\epsilon}^{1} \frac{1}{u}du = \lim_{\epsilon \mapsto 0} \frac{1}{e} \ln u|_{\epsilon}^{1} = \frac{-1}{e} \underbrace{\lim_{\epsilon \mapsto 0} \ln \epsilon}_{-\infty} = +\infty$$

So

$$\Gamma(0) = \infty$$

Using similar tricks,

$$\Gamma(a) = \infty \qquad a < 0$$

Generalize, for $n \in \mathbb{N}^{+}$

$$\Gamma(n+1) = n\Gamma(n) = ... = n(n-1)(a-2)...\Gamma(1) = n!$$

$\Gamma(\frac{1}{2})$

$$\Gamma(\frac{1}{2}) = \int_0^{\infty} e^{-u}u^{-\frac{1}{2}}du = \int_0^{\infty} e^{-u}\frac{du}{\sqrt{u}}$$

$\dfrac{du}{\sqrt{u}} = 2d\sqrt{u}$

$$= 2\int_0^{\infty} e^{-u}d\sqrt{u}$$

Let $\sqrt{u} = x \;\rightarrow \begin{cases} d\sqrt{u} = dx \\ u = x^2 \end{cases}$

$$= 2\int_0^{\infty} e^{-x^2}dx$$

Tricks on integration of $\exp(-x^2)$

$$= 2\left\{\int_0^{\infty} e^{-x^2}dx\right\}^{\frac{2}{2}} = 2\left\{\int_0^{\infty} e^{-x^2}dx \cdot \int_0^{\infty} e^{-y^2}dy\right\}^{\frac{1}{2}} = 2\left\{\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dxdy\right\}^{\frac{1}{2}}$$

Coordinate Transform

Rectangular $(x\,,\,y) \longrightarrow Polar(r\,,\,\theta)$ $\begin{cases} dxdy = rdrd\theta \\ x^2+y^2 = r^2 \end{cases}$ $\quad x, y\,\in [0+\infty] \longrightarrow \begin{cases} r \in [0,\infty] \\ \theta \in [0,\frac{\pi}{2}] \end{cases}$ First Quarda

$$= 2\left\{\int_0^{\infty}\int_0^{\pi/2} e^{-r^2}rdrd\theta\right\}^{\frac{1}{2}} rdr = \frac{1}{2}dr^2 = 2\left\{\frac{1}{2}\int_0^{\infty}\int_0^{\pi/2} e^{-r^2}dr^2d\theta\right\}^{\frac{1}{2}} = 2\sqrt{\frac{1}{2}\int_0^{\pi/2}[-e^{-r^2}]_0^{\infty}\,d\theta}$$

$$= 2\sqrt{\frac{1}{2}\int_0^{\pi/2} d\theta} = 2\sqrt{\frac{1}{2}\cdot\frac{\pi}{2}} = \sqrt{\pi}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

2 Poisson and Gamma Distributions

The Problem. A random variable X given Λ has the Poisson distribution. The parameter $\Lambda = \lambda$ has the Gamma distribution. Then what is the pdf of X ? What is $\mathbb{E}X$?

Poisson and Gamma Distributions

For Poisson distribution, the pdf is $f(x|\lambda) = \dfrac{e^{-\lambda}\lambda^x}{x!}$ where λ is a parameter

For Gamma distribution, the pdf is $f(x|\alpha,\beta) = \dfrac{\beta^\alpha \lambda^{\alpha-1}e^{-x\beta}}{\Gamma(\alpha)}$ where α, β are parameters and $\Gamma(x)$ is the Gamma function $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. Note that in some book β is replaced by $\dfrac{1}{\beta}$.

The derivation of $\mathbb{E}X$.

Since X given $\Lambda = \lambda$ is Poisson distributed , so $f(x|\lambda) = \dfrac{e^{-\lambda}\lambda^x}{x!}$

Since λ given α, β is Gamma distributed, so $f(\lambda|\alpha,\beta) = \dfrac{\beta^\alpha \lambda^{\alpha-1}e^{-\lambda\beta}}{\Gamma(\alpha)}$

Then, to find $\mathbb{E}X$, the following equation has to be applibed

$$\mathbb{E}X = \mathbb{E}\mathbb{E}(X|\Lambda)$$

Thus

$$\mathbb{E}X = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x|\lambda)\, f(\lambda|\alpha,\beta)\, d\lambda dx$$

Since $f(x|\lambda) = \dfrac{e^{-\lambda}\lambda^x}{x!}$ and $f(\lambda|\alpha,\beta) = \dfrac{\beta^\alpha \lambda^{\alpha-1}e^{-\lambda\beta}}{\Gamma(\alpha)}$

$$\mathbb{E}X = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\frac{e^{-\lambda}\lambda^x}{x!}\frac{\beta^\alpha \lambda^{\alpha-1}e^{-\lambda\beta}}{\Gamma(\alpha)}dxd\lambda$$

Factorize

$$\mathbb{E}X = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\frac{1}{x!}\frac{\beta^\alpha \lambda^{\alpha+x-1}e^{-\lambda(\beta+1)}}{\Gamma(\alpha)}dxd\lambda$$

To simplify this integral, one has to apply the following

$$\int_{-\infty}^{\infty}\frac{B^A}{\Gamma(A)}\lambda^{A-1}e^{-B\lambda}d\lambda = 1$$

First re-arrange the terms

$$\mathbb{E}X = \int_{-\infty}^{\infty} x\frac{\beta^\alpha}{\Gamma(\alpha)x!}\int_{-\infty}^{\infty}\lambda^{\alpha+x-1}e^{-\lambda(\beta+1)}dxd\lambda$$

Then

$$\mathbb{E}X = \int_{-\infty}^{\infty} x\frac{\beta^\alpha}{\Gamma(\alpha)x!}\int_{-\infty}^{\infty}\left\{\frac{(\beta+1)^{x+\alpha}}{(\beta+1)^{x+\alpha}}\cdot\frac{\Gamma(x+\alpha)}{\Gamma(x+\alpha)}\right\}\lambda^{\alpha+x-1}e^{-\lambda(\beta+1)}dxd\lambda$$

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x\frac{\beta^\alpha}{\Gamma(\alpha)x!}\frac{\Gamma(x+\alpha)}{(\beta+1)^{x+\alpha}}\left(\underbrace{\int_{-\infty}^{\infty}\frac{(\beta+1)^{x+\alpha}}{\Gamma(x+\alpha)}\lambda^{x+\alpha-1}e^{-(\beta+1)\lambda}d\lambda}_{1}\right)dx$$

Then

$$\mathbb{E}\left(X\right) = \int_{-\infty}^{\infty} x \underbrace{\frac{\Gamma\left(x+\alpha\right)}{\Gamma(\alpha)} \frac{1}{x!} \frac{\beta^{\alpha}}{\left(\beta+1\right)^{x+\alpha}}}_{f(x)} dx$$

$$\mathbb{E}\left(X\right) = \int_{-\infty}^{\infty} x \frac{\Gamma\left(x+\alpha\right)}{\Gamma(\alpha)} \frac{1}{x!} \left(\frac{\beta}{\beta+1}\right)^{\alpha} \frac{1}{\left(\beta+1\right)^{x}} dx = \left(\frac{\beta}{\beta+1}\right)^{\alpha} \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} \frac{\Gamma\left(x+\alpha\right)}{\Gamma(x)} \frac{1}{\left(\beta+1\right)^{x}} dx$$

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