Discrete Distributions : Bernoulli, Binomial and Poisson

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December 11, 2012

	Sample Space	PDF	μ	σ^2	MGF	CDF
Bernoulli	$X = \{1, 0\}$	$p^x(1-p)^{1-x}$	p	$p(1-p)$	$1 - p + pe^t$	$\begin{cases} 0 & x < 0 \\ p & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$
Binomial	$X = \{\,1, 2, 3, ..., n\,\}$	$\binom{n}{x} p^x(1-p)^{n-x}$	np	$np(1-p)$	$(1 - p + pe^t)^n$	$\sum_{x=0}^{n} \binom{n}{x} p^x(1-p)^{n-x}$
Poisson	$n \to \infty$	$\dfrac{\mu^x e^{-\mu}}{x!}$	μ	μ	$e^{\mu\left(e^t - 1\right)}$	$e^{-\mu} \sum_{x=0}^{n} \dfrac{\mu^x}{x!}$

1 Bernoulli Distributions

1.1 PDF

$$Ber = \{Success\,,\,Fail\} = \{1, 0\}$$

$$PDF = Ber(x; p) = p^x(1-p)^{1-x} \qquad \text{where } k = 1\,,\,0$$

Illustration :

- When $x = 1$ (success) , $Ber(1; p) = p = \text{Prob}(Success)$
- When $x = 0$ (fail) , $Ber(0; p) = (1 - p) = \text{Prob}(fail)$
- Summation of PDF = 1 : $p + (1 - p) = 1$

Since Bermoulli Random Variable is a 2-membered sample space, so to find the μ and σ^2, just diectly use the definitions

1.2 μ_{Ber}

$$\mu_{Ber} = E(X_{Ber}) = \sum xf(x) = Success \times \text{Prob}(Success) + Fail \times \text{Prob}(Fail) = 1 \times p + 0 \times (1 - p) = p$$

1.3 σ^2_{Ber}

$$\sigma^2_{Ber} = Var(X_{Ber}) = \sum (x_i - \mu)^2 f(x) = (Success - Mean)^2 Prob(Success) + (Fail - Mean)^2 Prob(Fail)$$

$$= (1 - p)^2 p + (0 - p)^2(1 - p) = (1 - p)p$$

1.4 MFG

$$M_{Ber}(t) = E\left[e^{t(Ber)}\right] = \sum e^{tBer} Ber(x; p) = e^{t\times 1}p + e^{t\times 0}(1 - p) = pe^t + 1 - p$$

1.5 CDF

$$CDF_{Ber} = \begin{cases} 0 & x < 0 \\ p & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

2 Binomial Distribution

2.1 Binomial Coefficient Review

$$x\binom{n}{x} = n\binom{n-1}{x-1}$$

2.2 PDF

Binomial Distribution is the generalization of the Bernoulli Distribution

$$Bin(x;n,p) = \binom{n}{x} p^x (1-p)^{n-x}$$

Illustration

- When 1 trial ($n = 1$), success ($x = 1$) , then $Bin(1;1,p) = p = \text{Prob}(Success) = \text{Prob}(One\ Success)$
- When 1 trial ($n = 1$), fail ($x = 0$) , then $Bin(0;1,p) = 1-p = \text{Prob}(Fail) = \text{Prob}(One\ Fail)$
- When 2 trial ($n = 2$) , one fail one success ($x = 1$), then $Bin(1;2,p) = 2p(1-p) = 2 \times \text{Prob}(Success) \times \text{Prob}(Fail)$, true, since one fail one success can be SF or FS (S=Success, F=Fail)
- When 3 trial ($n = 3$) , two success ($x = 1$, that means 1 fail) , then $Bin(2;3,p) = \binom{3}{2} p^2(1-p)^{3-2} = 3 \times \text{Prob}(Success) \times \text{Prob}(Success) \times \text{Prob}(Fail)$, also ture, since there are 3 combination of 2-success-1-fail in a 3 trial : SSF, SFS, FSS.
- Therefore the term $\binom{n}{x}$ in $\binom{n}{x} p^x(1-p)^{n-x}$ means the number of all possible combinations of the required outcome.
- Summation of PDF = 1 : $\sum_{x=0}^{n} \binom{n}{x} p^x(1-p)^{n-x} = (1 + (1-p))^n = 1^n = 1$

2.3 μ_{Bin}

$$\begin{aligned}
\mu_{Bin} &= E(X_{Bin}) = \sum x f(x) = \sum x \binom{n}{x} p^x(1-p)^{n-x} \\
&= \sum x \binom{n}{x} p^x(1-p)^{n-x} = \sum x \frac{n!}{(n-x)!x!} p^x(1-p)^{n-x} \\
&= \sum \frac{n(n-1)!}{(n-x)!(x-1)!} p^{x-1} p (1-p)^{n-x} \\
&= np \underbrace{\sum \frac{(n-1)!}{(n-x)!(x-1)!} p^{x-1}(1-p)^{(n-1)-(x-1)}}_{\text{Binomial Expansion}} \\
&= np\,(p + (1-p))^{n-1} = np
\end{aligned}$$

2.4 σ^2_{Bin}

$$\sigma^2 = E[X^2] - (E[X])^2$$

Now consider the second order moment

$$E[X^2] = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} = np \sum_{x=1}^{n} x \binom{n-1}{x-1} p^{x-1} (1-p)^{(n-1)-(x-1)}$$

To avoid very long equation, let $x - 1 = y$ so , $x = y + 1$ and let $1 - p = q$

$$np \sum_{x-1=1-1}^{n-1} (y+1) \binom{n-1}{y} p^y q^{(n-1)-y} = np \sum_{y=0}^{n-1} (y+1) \binom{n-1}{y} p^y q^{(n-1)-y}$$

Now also let $n - 1 = m$

$$np \sum_{y=0}^{m} (y+1) \binom{m}{y} p^y q^{m-y} = np \left(\sum_{y=0}^{m} y \binom{m}{y} p^y q^{m-y} + \underbrace{\sum_{y=0}^{m} \binom{m}{y} p^y q^{m-y}}_{\text{Binomial Expansion}} \right)$$

$$np \left(\sum_{y=0}^{m} m \binom{m-1}{y-1} p^y q^{m-y} + \underbrace{(p+q)^m}_{1} \right) = np \left(\underbrace{mp \sum_{y=0}^{m} \binom{m-1}{y-1} p^{y-1} q^{(m-1)-(y-1)}}_{\text{Binomial Expansion}} + \underbrace{(p+q)^m}_{1} \right)$$

$$E[X^2] = np \left(\underbrace{m}_{n-1} p \underbrace{(p+q)^{m-1}}_{1} + 1 \right) = np + n^2p^2 - np^2$$

Thus

$$\sigma^2 = E[X^2] - (E[X])^2 = \left(np + n^2p^2 - np^2\right) - (np)^2 = np - np^2 = np(1-p)$$

2.5 MGF

$$M_{Bin}(t) = E\left[e^{t(Bin)}\right] = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=0}^{n} \binom{n}{x} \left(pe^t\right)^x (1-p)^{n-x} = \left[pe^t + (1-p)\right]^n$$

2.6 CDF

$$CDF = \begin{cases} \sum_{x=0}^{n} \binom{n}{x} p^x (1-q)^{n-x} & n \in \mathbb{Z} \\ \sum_{x=0}^{\lfloor n \rfloor} \binom{\lfloor n \rfloor}{x} p^x (1-q)^{\lfloor n \rfloor - x} & n \in \mathbb{R} \end{cases}$$

Remark. The CDF of Binomial Random Variable can also be expressed in terms of Beta Function.

3 Poisson Distribution

3.1 Derivation of the PDF

Poisson Distribution is the generalization of the Binomial Distribution

$$\lim_{n\to\infty}\binom{n}{x}p^x(1-p)^{n-x}=\lim_{n\to\infty}\frac{n!}{(n-x)!x!}p^x(1-p)^{n-x}$$

Since $\mu_{Bin}=np$, so $p=\dfrac{\mu}{n}$

$$\lim_{n\to\infty}\frac{n!}{(n-x)!x!}\left(\frac{\mu}{n}\right)^x\left(1-\frac{\mu}{n}\right)^{n-x}$$

1. Take the term that do not have n out of the limit
2. Make $(1-A)$ term be $(1+(-A))$

$$\lim_{n\to\infty}\frac{n(n-1)...(n-x+1)}{x!}\frac{\mu^x}{n^x}\left(1-\frac{\mu}{n}\right)^n\left(1-\frac{\mu}{n}\right)^{-x}=\frac{\mu^x}{x!}\lim_{n\to\infty}\frac{n(n-1)...(n-x+1)}{n^x}\left(1+\frac{-\mu}{n}\right)^n\left(1+\frac{-\mu}{n}\right)^{-x}$$

Notice that $\underbrace{n(n-1)...(n-x+1)}_{x-term}$ and n^x also has $x-$term

$$\frac{\mu^x}{x!}\lim_{n\to\infty}\left(\frac{n}{n}\cdot\frac{n-1}{n}...\frac{n-x+1}{n}\right)\left(1+\frac{-\mu}{n}\right)^n\left(1+\frac{-\mu}{n}\right)^{-x}$$

To form the $e^{-\mu}$:

$$\left(1+\frac{-\mu}{n}\right)^n=\left(1+\frac{1}{\frac{n}{-\mu}}\right)^{\frac{-\mu n}{-\mu}}=\left(1+\frac{1}{\left(\frac{n}{-\mu}\right)}\right)^{-\mu\left(\frac{n}{-\mu}\right)}$$

i.e.

$$\frac{\mu^x}{x!}\lim_{n\to\infty}\left(\frac{n}{n}\cdot\frac{n-1}{n}...\frac{n-x+1}{n}\right)\left(1+\frac{1}{\left(\frac{n}{-\mu}\right)}\right)^{-\mu\left(\frac{n}{-\mu}\right)}\left(1+\frac{-\mu}{n}\right)^{-x}$$

Take the limit

$$\frac{\mu^x}{x!}\left[\lim_{n\to\infty}\underbrace{\left(\frac{n}{n}\cdot\frac{n-1}{n}...\frac{n-x+1}{n}\right)}_{\text{All are 1}}\right]\left[\lim_{n\to\infty}\left(1+\frac{1}{\left(\frac{n}{-\mu}\right)}\right)^{-\mu\left(\frac{n}{-\mu}\right)}\right]\left[\underbrace{\lim_{n\to\infty}\left(1+\frac{-\mu}{n}\right)^{-x}}_{1}\right]$$

$$\frac{\mu^x}{x!}e^{-\mu}$$

i.e.

$$Poi(x;\mu)=\frac{\mu^x e^{-\mu}}{x!}$$

Remark. n , the number of trial, or sample size, is droped out.

When Poisson Distribution is used to approximate Binomial Distribution (n is large)

$$Poi(x;n,p)=\frac{(np)^x e^{-np}}{x!}$$

Illustration

- Sum of PDF $=1$: $\sum_{x=0}^{n}\dfrac{\mu^x e^{-\mu}}{x!}=e^{-\mu}\sum_{x=0}^{n}\dfrac{\mu^x}{x!}=e^{-\mu}\left(\dfrac{1}{0!}+\dfrac{\mu}{1!}+\dfrac{\mu^2}{2!}+...\right)=e^{-\mu}e^{+\mu}=1$

3.2 μ_{Poi}

$$\mu_{Poi} = E[X] = \sum xPoi(x;\mu) = \sum_{x=0}^{n} x \cdot \frac{\mu^x e^{-\mu}}{x!} = e^{-\mu} \sum_{x=1}^{n} \frac{\mu^x}{(x-1)!} = \mu e^{-\mu} \sum_{x-1=1-1}^{n-1} \frac{\mu^{x-1}}{(x-1)!}$$

Let $y = x - 1$, $n - 1 = m$

$$\mu e^{-\mu} \sum_{y=0}^{m} \frac{\mu^y}{y!} = \mu e^{-\mu} e^{+\mu} = \mu$$

3.3 σ^2_{Poi}

$$\sigma^2 = E[X^2] - (E[X])^2$$

$$E[X^2] = \sum_{x=0}^{n} x^2 Poi(x;\mu) = \sum_{x=0}^{n} x^2 \frac{\mu^x e^{-\mu}}{x!} = e^{-\mu} \sum_{x-1=1-1}^{n-1} \frac{x\mu^x}{(x-1)!}$$

Let $x - 1 = y$, $n - 1 = m$

$$e^{-\mu} \sum_{y=0}^{m} \frac{(y+1)\mu^{y+1}}{y!} = e^{-\mu} \sum_{y=0}^{m} \frac{y\mu^{y+1}}{y!} + e^{-\mu} \sum_{y=0}^{m} \frac{\mu^{y+1}}{y!} = \mu e^{-\mu} \sum_{y=1}^{m} \frac{\mu^y}{(y-1)!} + \mu e^{-\mu} \sum_{y=0}^{m} \frac{\mu^y}{y!}$$

$$= \mu e^{-\mu} \sum_{y=1}^{m} \frac{\mu^y}{(y-1)!} + \mu e^{-\mu} e^{\mu} = \mu e^{-\mu} \sum_{y-1=1-1}^{m-1} \frac{\mu^y}{(y-1)!} + \mu$$

Let $y - 1 = k$, $m - 1 = r$

$$\mu e^{-\mu} \sum_{k=0}^{r} \frac{\mu^{k+1}}{k!} + \mu = \mu^2 e^{-\mu} \sum_{k=0}^{r} \frac{\mu^k}{k!} + \mu = \mu^2 e^{-\mu} e^{\mu} + \mu = \mu^2 + \mu$$

Therefore

$$\sigma^2_{Poi} = E[X^2] - (E[X])^2 = \left(\mu^2 + \mu\right) - (\mu)^2 = \mu$$

3.4 MGF

$$M_{Poi}(t) = E\left[e^{t(Poi)}\right] = \sum_{x=0}^{n} e^{tx} \frac{e^{-\mu}\mu^x}{x!} = e^{-\mu} \sum_{x=0}^{n} \frac{(e^t\mu)^x}{x!}$$

Since

$$e^\theta = \frac{1}{0!} + \frac{\theta}{1!} + \frac{\theta^2}{2!} + ... = \sum \frac{\theta^x}{x!}$$

Thus

$$M_{Poi}(t) = e^{-\mu} \sum_{x=0}^{n} \frac{(e^t\mu)^x}{x!} = e^{-\mu} \exp\left(e^t\mu\right) = \exp\left(e^t\mu - \mu\right) = \exp\left(\mu\left(e^t - 1\right)\right)$$

3.5 CDF

$$CDF = \sum_{x=0}^{n} \frac{e^{-\mu}\mu^x}{x!} = e^{-\mu} \sum_{x=0}^{n} \frac{\mu^x}{x!}$$

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