# Continuous Distributions : Uniform, Expotential and Gaussian

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| | Sample Space | $PDF$ | $\mu$ | $\sigma^2$ | MGF | CDF |
|---|---|---|---|---|---|---|
| Uniform | $x \in [a,b]$ | $\dfrac{1}{b-a}$ | $\dfrac{a+b}{2}$ | $\dfrac{(b-a)^2}{12}$ | $\begin{cases} \dfrac{e^{tb}-e^{ta}}{t(b-a)} & t \neq 0 \\ 1 & t=0 \end{cases}$ | $\begin{cases} 0 & a < x \\ \dfrac{x-a}{b-a} & x \in [a,b) \\ 1 & x \geq b \end{cases}$ |
| Exponential | $x \in [0,\infty)$ | $\dfrac{e^{-\frac{x}{\theta}}}{\theta}$ | $\theta$ | $\theta^2$ | $\dfrac{1}{1-\theta t}$ | $1-e^{-\frac{x}{\theta}}$ |
| Gaussian | $x \in \mathbb{R}$ | $\dfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ | $\mu$ | $\sigma^2$ | $\exp\left(\mu t + \dfrac{1}{2}\sigma^2 t^2\right)$ | $\dfrac{1}{2}\left[1+\text{erf}\left(\dfrac{x-\mu}{\sqrt{2\sigma^2}}\right)\right]$ |

# 1 Uniform

## 1.1 PDF

$$Uni(x) = \frac{1}{b-a} = \text{const.}$$

Illustration :

- Summation of PDF = 1 : $\int_a^b \frac{1}{b-a} dx = 1$

## 1.2 $\mu_{Uni}$

$$\mu_{Uni} = E(X_{Uni}) = \int_a^b x f(x) dx = \int_a^b \frac{x}{b-a} dx = \frac{a+b}{2}$$

## 1.3 $\sigma^2_{Uni}$

$$\sigma^2_{Uni} = Var(X_{Uni}) = \int_a^b x^2 f(x) dx - \mu^2_{Uni} = \frac{1}{b-a}\int_a^b x^2 dx - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right) - \frac{a^2+2ab+b^2}{4} = \frac{4a^2+4b^2+4ab-3a^2-6ab-3b^2}{12} = \frac{(a-b)^2}{12}$$

## 1.4 MFG

$$M_{Uni}(t) = E\left[e^{t(Uni)}\right] = \begin{cases} t \neq 0 \ : \ \int_a^b e^{tx}\dfrac{dx}{b-a} = \dfrac{e^{tb}-e^{ta}}{t(b-a)} \\ \\ t = 0 \ : \ \int_a^b \dfrac{dx}{b-a} = 1 \end{cases}$$

### 1.5 CDF

$$\begin{cases} 0 & a < x \\ \dfrac{x-a}{b-a} & x \in [a,b) \\ 1 & x \geq b \end{cases}$$

## 2 Exponential

### 2.1 PDF

$$E\left(x;\theta\right) = \frac{1}{\theta}e^{-\frac{x}{\theta}} \quad or \quad \lambda e^{-\lambda x} \qquad \frac{1}{\theta} = \lambda$$

- Summation of PDF = 1 : $\int_0^\infty \frac{1}{\theta}e^{-\frac{x}{\theta}}dx = -\int_0^\infty e^{-\frac{x}{\theta}}\left(d\frac{-x}{\theta}\right) = -\left(\exp\frac{-x}{\theta}\right)_0^\infty = 1$

### 2.2 $\mu_{Exp}$

$$\mu_{Exp} = E(X_{Exp}) = \int_0^\infty x\frac{1}{\theta}e^{-\frac{x}{\theta}}dx = -\int_0^\infty xe^{-\frac{x}{\theta}}\left(d\frac{-x}{\theta}\right) = -\int_0^\infty x\left(de^{-\frac{x}{\theta}}\right)$$

$$= \underbrace{-\left(xe^{-\frac{x}{\theta}}\right)_0^\infty}_{0} + \int_0^\infty e^{-\frac{x}{\theta}}dx = -\theta\underbrace{\int_0^\infty e^{-\frac{x}{\theta}}d\frac{-x}{\theta}}_{-1} = \theta$$

### 2.3 $\sigma^2_{Exp}$

$$\sigma^2 = E[X^2] - (E[X])^2$$

Now consider the second order moment

$$E[X^2] = \int_0^\infty x^2\frac{1}{\theta}e^{-\frac{x}{\theta}}dx = -\int_0^\infty x^2e^{-\frac{x}{\theta}}\left(d\frac{-x}{\theta}\right) = -\int_0^\infty x^2de^{-\frac{x}{\theta}}$$

$$= \underbrace{-\left(x^2e^{-\frac{x}{\theta}}\right)_0^\infty}_{0} + 2\int_0^\infty xe^{-\frac{x}{\theta}}dx = 2\theta\underbrace{\int_0^\infty \frac{1}{\theta}xe^{-\frac{x}{\theta}}dx}_{E(X)} = 2\theta^2$$

Thus

$$\sigma^2 = E[X^2] - (E[X])^2 = \left(2\theta^2\right) - \left(\theta^2\right) = \theta^2$$

### 2.4 MGF

$$M_{Exp}(t) = E\left[e^{t(Exp)}\right] = \int_0^\infty e^{tx}\frac{1}{\theta}e^{-\frac{x}{\theta}}dx = \frac{1}{\theta}\int_0^\infty \exp\left(-x\left(\frac{1}{\theta}-t\right)\right)dx$$

$$= \frac{1}{-\left(\frac{1}{\theta}-t\right)\theta}\underbrace{\int_0^\infty d\exp\left(-x\left(\frac{1}{\theta}-t\right)\right)}_{-1} = \frac{1}{1-\theta t}$$

### 2.5 CDF

$$CDF = \int_0^x \frac{1}{\theta}e^{-\frac{x}{\theta}}dx = -\int_0^x e^{-\frac{x}{\theta}}d\frac{-x}{\theta} = -\left(e^{-\frac{x}{\theta}}\right)_0^x = 1 - e^{-\frac{x}{\theta}}$$

# 3 Gaussian

## 3.1 PDF

$$N(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

## 3.2 Integration of $e^{-x^2}$ , Review of Double Integral and Jacobian

$$\int_{-\infty}^{\infty} e^{-kx^2} dx = \sqrt{\int_{-\infty}^{\infty} e^{-kx^2} dx \int_{-\infty}^{\infty} e^{-kx^2} dx} = \left(\int_{-\infty}^{\infty} e^{-kx^2} dx \int_{-\infty}^{\infty} e^{-ky^2} dy\right)^{\frac{1}{2}} = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-k(x^2+y^2)} dxdy\right)^{\frac{1}{2}}$$

$x = r\cos\theta$ , $y = r\sin\theta$ , so the partial derivatives are : $x_r = \cos\theta$ , $x_\theta = -r\sin\theta$ , $y_r = \sin\theta$ , $y_\theta = r\cos\theta$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-k(x^2+y^2)} dxdy = \iint e^{-kr^2} J dr d\theta \qquad J = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Therefore

$$\int_{-\infty}^{\infty} e^{-kx^2} dx = \left(\int_0^{2\pi}\int_0^{\infty} e^{-kr^2} r dr d\theta\right)^{\frac{1}{2}} = \left(\underbrace{\int_0^{2\pi} d\theta}_{2\pi} \frac{-1}{2k} \underbrace{\int_0^{\infty} e^{-kr^2} d\left(-kr^2\right)}_{-1}\right)^{\frac{1}{2}} = \sqrt{\frac{\pi}{k}}$$

## 3.3 Sum of PDF = 1

$$\int_{-\infty}^{\infty}\left(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right) dx$$

Let $t = \frac{x-\mu}{\sigma}$ , so $dx = \sigma dt$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt$$

Apply pervious part with $k = \frac{1}{2}$ , the integral is thus

$$\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi}{\frac{1}{2}}} = 1$$

## 3.4 $\mu$

$$\mu = E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Let $t = \frac{x-\mu}{\sigma}$ , $dx = \sigma dt$ , $x = t\sigma + \mu$

$$\int_{-\infty}^{\infty} (t\sigma + \mu)\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = \int_{-\infty}^{\infty} \frac{t\sigma}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt + \int_{-\infty}^{\infty} \frac{\mu}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

The first integral is zero, since $te^{-\frac{t^2}{2}}$ is an odd function

The second integral is the type in previous part with $k = \frac{1}{2}$ , thus the mean of Gaussian Distribution is

$$\mu \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt}_{1} = \mu$$

### 3.5 $\sigma^2$

$$\sigma^2 = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Let $t = \dfrac{x-\mu}{\sigma}, dx = \sigma dt, x = t\sigma + \mu$

$$\int_{-\infty}^{\infty} \left(t^2\sigma^2 + 2t\sigma\mu + \mu^2\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$\underbrace{\int_{-\infty}^{\infty} \frac{t^2\sigma^2}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt}_{\text{By-parts}} + \underbrace{\int_{-\infty}^{\infty} \frac{2t\sigma\mu}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt}_{\text{Zero , odd function}} + \underbrace{\int_{-\infty}^{\infty} \frac{\mu^2}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt}_{\mu^2}$$

$$\frac{\sigma^2}{2} \int_{-\infty}^{\infty} \frac{t\sigma^2}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt^2 + \mu^2 = -\sigma^2 \int_{-\infty}^{\infty} \frac{t}{\sqrt{2\pi}} de^{-\frac{t^2}{2}} + \mu^2 = \underbrace{\left[-\sigma^2 \frac{t}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\right]_{-\infty}^{\infty}}_{0} + \sigma^2 \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt}_{1} + \mu^2 = \sigma^2 + \mu^2$$

Therefore

$$Var(X) = E[X^2] - \mu^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$$

*Remark.* $\left[-\sigma^2 \frac{t}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\right]_{-\infty}^{\infty} = \frac{-\sigma^2}{\sqrt{2\pi}} \left[\frac{t}{e^{-\frac{t^2}{2}}}\right]_{-\infty}^{\infty} = \frac{-\sigma^2}{\sqrt{2\pi}} \left[\lim_{t\to\infty} \frac{t}{e^{-\frac{t^2}{2}}} - \lim_{t\to-\infty} \frac{t}{e^{-\frac{t^2}{2}}}\right]$

$$\overset{L'Hosptial}{=} \frac{-\sigma^2}{\sqrt{2\pi}} \left[\lim_{t\to\infty} \frac{1}{-te^{-\frac{t^2}{2}}} - \lim_{t\to-\infty} \frac{1}{-te^{-\frac{t^2}{2}}}\right] = 0$$

### 3.6 MGF

$$M(t) = E\left[e^{tx}\right] = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[tx - \frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

Let $z = \dfrac{x-\mu}{\sigma}$ , $dx = \sigma dz$ , $x = z\sigma + \mu$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[t\sigma z + t\mu - \frac{z^2}{2}\right] dz$$

$$= (\exp t\mu) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp -\left[\frac{z^2}{2} - t\sigma z\right] dz$$

$$= (\exp t\mu) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp -\left[\frac{z^2}{2} - t\sigma z + \frac{t^2\sigma^2}{2} - \frac{t^2\sigma^2}{2}\right] dz$$

$$= (\exp t\mu) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp -\left[\frac{(z-t\sigma)^2}{2} - \frac{t^2\sigma^2}{2}\right] dz$$

$$= \exp\left(t\mu + \frac{t^2\sigma^2}{2}\right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp - \left[\frac{(z - t\sigma)^2}{2}\right] dz$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp - \left[\frac{(z - t\sigma)^2}{2}\right] dz = 1 \qquad \text{By applying previous integral with } k = \frac{1}{2}$$

Therefore

$$M_X(t) = \exp\left(t\mu + \frac{t^2\sigma^2}{2}\right)$$

### 3.7 CDF

$$CDF = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Which is the standard Error Function

$$\frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x - \mu}{\sqrt{2\sigma^2}}\right)\right]$$

$$-END-$$