Univariate Gaussian Distribution Derivation

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This derivation comes from Dr. Dan Teague, this document is a short notes of Dr. Dan Teagues's Derivation.

This proof is very good, and easy to understand.

Prerequisite (To junior students)

To understand this derivation , you need to know

- Statistics : Variance, First order moment, Expectation
- Vairable Separable ODE
- How to do $\int e^{-Ax^2} dx$ by Jacobian
- L'Hospital Rule & Squeeze Theorem

Derivation

Consider aiming at origin of Cartesian plane with darts.

Assumption

1. Deviation of darts does not depend on orientation of coordinate system
2. Deviations in orthogonal directions are independent. This means if we deviate a lot in one direction the probabilities of other direction are not influenced.
3. Large deviations are less likely than small deviations.

Start

Consider throwing an dart,

$$P\Bigg(\text{dart falls in interval } [x+\Delta x]\Bigg)_{\Delta x\,\text{infinitesimal}} = \int_x^{x+\Delta x} p(x)dx \approx p(x)\Delta x$$

Similarly, for vertical direction,

$$P\Bigg(\text{dart falls in interval } [y+\Delta y]\Bigg)_{\Delta y\,\text{infinitesimal}} = \int_y^{y+\Delta y} p(y)dy \approx p(y)\Delta y$$

Under independence assumption of perpendicular direction, region probability is

$$P(\text{area}) = p(x)p(y)\Delta y\Delta x$$

If not aim at origin $(0,0)$ but in $(\mu\,,\,\mu)$, the translated$P(E)$ change to

$$p(x+\mu)p(y+\mu)\Delta y\Delta x$$

By assumption that orientation does not matter, any region r unit away from origin with area $\Delta x\Delta y$ has same $P(E)$

$$p(x+\mu)p(y+\mu)\Delta y\Delta x = g(r)\Delta x\Delta y$$

i.e.

$$g(r) = p(x+\mu)p(y+\mu)$$

$g(r)$ is not depended on θ , as every $P(E)$ is same, so function is only dependent on radius

Using polar coordinate, $x = r\text{cos}\theta$, $y = r\text{sin}\theta$

$$\frac{d}{d\theta}g(r) = p(x+\mu)\frac{d}{d\theta}p(y+\mu) + p(y+\mu)\frac{d}{d\theta}p(x+\mu)$$

$$= p(x+\mu)\frac{dp(y+\mu)}{d(y+\mu)}\frac{d(y+\mu)}{d\theta} + p(y+\mu)\frac{dp(x+\mu)}{d(x+\mu)}\frac{d(x+\mu)}{d\theta}$$

Recall, μ is constant

$$= p(x+\mu)p'(y+\mu)r\text{cos}\theta + p(y+\mu)p'(x+\mu)(-r\text{sin}\theta)$$

Recall, $g(r)$ is independent of θ, $\dfrac{dg(r)}{d\theta} = 0$

$$0 = p(x+\mu)p'(y+\mu)r\text{cos}\theta - p(y+\mu)p'(x+\mu)r\text{sin}\theta$$

In rectangular coordinate,

$$p(x+\mu)p'(y+\mu)x = p(y+\mu)p'(x+\mu)y$$

Which is a ODE that is separable

$$\frac{p'(y+\mu)}{p(y+\mu)y} = \frac{p'(x+\mu)}{p(x+\mu)x}\ \forall x,y \in \mathbb{R}$$

Solving using separable,

$$\frac{p'(y+\mu)}{p(y+\mu)y} = \frac{p'(x+\mu)}{p(x+\mu)x} = C$$

$$\frac{p'(y+\mu)}{p(y+\mu)y} = C\ \ \frac{p'(x+\mu)}{p(x+\mu)x} = C$$

$$\frac{\frac{dp(y+\mu)}{d(y+\mu)}}{p(y+\mu)y} = C \iff \int\frac{dp(y+\mu)}{p(y+\mu)} = \int Cyd(y+\mu)$$

$$\ln p(y+\mu) = \frac{C_1}{2}y^2 + C_2$$

$$p(y+\mu) = A\ \exp\left[\frac{C_1}{2}y^2\right]\ \ p(x+\mu) = B\ \exp\left[\frac{C_1}{2}x^2\right]$$

i.e.

$$p(y+\mu) = Ae^{\frac{C}{2}y^2}\ \ p(x+\mu) = Be^{\frac{C}{2}x^2}$$

By the assumption of large deviations are less likely, $C < 0$ (must be negative) s.t. **probability density decreases when deviation becomes larger**.

$$p(y) = Ae^{-\frac{C}{2}(y-\mu)^2}\ \ p(x) = Be^{-\frac{C}{2}(x-\mu)^2}$$

One fundamental properties of probability distribution integral : Result must be equal to one. So we can use this to solve for the constant.

$$A\int_{\mathbb{R}} e^{-\frac{C}{2}(y-\mu)^2}dy = 1$$

$$\int_{-\infty}^{\infty} e^{-\frac{C}{2}(y-\mu)^2}dy = \frac{1}{A}$$

Let $t = y - \mu\ dy = dt$

$$\int_{-\infty}^{\infty} e^{-\frac{C}{2}t^2}dt = \frac{1}{A}$$

The integrand $\exp\left(-\frac{C}{2}t^2\right)$ is an **even function**

$$\int_{0}^{\infty} e^{-\frac{C}{2}t^2}dt = \frac{1}{2A}$$

Tricks of integrate exp $(-kt^2)$ $\quad Re(k) > 0$

$$\int_{0}^{\infty} e^{-kt^2}dt = \sqrt{\frac{\pi}{4k}}$$

Now $k = \frac{C}{2}$

$$\therefore \sqrt{\frac{\pi}{4\frac{C}{2}}} = \frac{1}{2A}$$

$$A = \pm\sqrt{\frac{C}{2\pi}}$$

We want real constant, so complex result is ignored. Then A can be real positive or real negative, for real negative :

$$p(x)_{x=\mu} = -\sqrt{\frac{C}{2\pi}}e^{-\frac{C}{2}(x-\mu)^2}\Big|_{x=\mu} = -\sqrt{\frac{C}{2\pi}} < 0$$

Propability can not be smaller than zero, so the constant should be a positive constant

$$0 \le \quad p(x) = \sqrt{\frac{C}{2\pi}} e^{-\frac{C}{2}(x-\mu)^2} \quad \le 1$$

Then find constant C by using expected value

$$E[x] = \sqrt{\frac{C}{2\pi}} \int_{\mathbb{R}} x \cdot e^{-\frac{C}{2}(x-\mu)^2} dx$$

Solve it by substitution, let $t = x - \mu$, $dx = dt$

$$= \sqrt{\frac{C}{2\pi}} \int_{\mathbb{R}} (t+\mu) e^{-\frac{C}{2}t^2} dt = \sqrt{\frac{C}{2\pi}} \int_{\mathbb{R}} t e^{-\frac{C}{2}t^2} dt + \mu \sqrt{\frac{C}{2\pi}} \int_{\mathbb{R}} e^{-\frac{C}{2}t^2} dt$$

$$\sqrt{\frac{C}{2\pi}} \int_{\mathbb{R}} t e^{-\frac{C}{2}t^2} dt = 0 \text{ (odd} \times \text{even} = \text{odd}\,, \int_{-a}^{a} \text{odd} = 0)$$

$$\mu \sqrt{\frac{C}{2\pi}} \int_{\mathbb{R}} e^{-\frac{C}{2}t^2} dt = \mu \sqrt{\frac{C}{2\pi}} \cdot \sqrt{\frac{\pi}{2C}} = \mu$$

$$E[x] = \mu$$

Recall, variance σ^2 of this distribution

$$(\text{Var}x)^2 = \sigma^2 = E\left[(x - E[x])^2\right] = E\left[(x-\mu)^2\right]$$

$$= E[x^2 + \mu^2 - 2x\mu] = E[x^2] + \mu^2 - 2\mu E[x]$$

$$= E[x^2] + \mu^2 - 2\mu^2 = E[x^2] - \mu^2 = E[x^2] - [E[x]]^2$$

i.e.

$$(\text{Var}x)^2 = \sigma^2 = E[x^2] - \mu^2$$

$$\sigma^2 + \mu^2 = E[x^2] = \sqrt{\frac{C}{2\pi}} \int_{\mathbb{R}} x^2 \cdot e^{-\frac{C}{2}(x-\mu)^2} dx$$

Again, let $x - \mu = t$

$$\sigma^2 + \mu^2 = \sqrt{\frac{C}{2\pi}} \int_{-\infty}^{\infty} (t+\mu)^2 e^{-\frac{C}{2}t^2} dt$$

$$= \sqrt{\frac{C}{2\pi}} \int_{-\infty}^{\infty} t^2 e^{-\frac{C}{2}t^2} dt + 2\mu \sqrt{\frac{C}{2\pi}} \int_{-\infty}^{\infty} t e^{-\frac{C}{2}t^2} dt + \mu^2 \sqrt{\frac{C}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{C}{2}t^2} dt$$

Middle integral

$$\int_{\mathbb{R}} te^{-\frac{C}{2}t^2}dt = 0 \text{ (odd} \times \text{even} = \text{odd}\,, \int_{-a}^{a} \text{odd} = 0)$$

$$\int_{\mathbb{R}} e^{-\frac{C}{2}t^2}dt = \sqrt{\frac{\pi}{2C}} = \mu$$

$$\sigma^2 + \mu^2 = \sqrt{\frac{C}{2\pi}}\int_{-\infty}^{\infty} t^2 e^{-\frac{C}{2}t^2}dt + \mu^2\sqrt{\frac{C}{2\pi}} \cdot \sqrt{\frac{\pi}{2C}}$$

$$\sigma^2 = \sqrt{\frac{C}{2\pi}}\int_{-\infty}^{\infty} t^2 e^{-\frac{C}{2}t^2}dt$$

Now find the integral by parts

$$\int_{-\infty}^{\infty} t^2 e^{-\frac{C}{2}t^2}dt = 2\int_{0}^{\infty} t^2 e^{-\frac{C}{2}t^2}dt \text{ (Even)}$$

$$\text{Let} \begin{cases} \alpha = t & d\alpha = dt \ \frac{d\alpha}{dt} = 1 \\ d\beta = dte^{-\frac{C}{2}t^2} \ \frac{d\beta}{dt} = te^{-\frac{C}{2}t^2} & \beta = \int te^{-\frac{C}{2}t^2}dt = \frac{1}{2}\int e^{-\frac{C}{2}t^2}dt^2 = \frac{-1}{C}e^{-\frac{C}{2}t^2} \end{cases}$$

$$\int t(te^{-\frac{C}{2}t^2})dt = \int \alpha(\beta' dt) = \beta\alpha - \int \beta d\alpha$$

$$\sigma^2 = \sqrt{\frac{C}{2\pi}}\left\{\left[\frac{-t}{C}e^{-\frac{C}{2}t^2}\right]_0^{\infty} - \int_{-\infty}^{\infty} \frac{-1}{C}e^{-\frac{C}{2}t^2}dt\right\} = \sqrt{\frac{C}{2\pi}}\left\{\left[\frac{-t}{C}e^{-\frac{C}{2}t^2}\right]_0^{\infty} + \frac{1}{C}\int_{-\infty}^{\infty} e^{-\frac{C}{2}t^2}dt\right\}$$

$$= \sqrt{\frac{C}{2\pi}}\left[\frac{-t}{C}e^{-\frac{C}{2}t^2}\right]_0^{\infty} + \sqrt{\frac{C}{2\pi}} \cdot \frac{1}{C}\sqrt{\frac{\pi}{2C}} = \sqrt{\frac{C}{2\pi}}\left[\frac{-t}{C}e^{-\frac{C}{2}t^2}\right]_0^{\infty} + \frac{1}{C}$$

$$\sigma^2 = \sqrt{\frac{C}{2\pi}} \cdot \frac{1}{C}\lim_{t\to\infty}\frac{t}{e^{\frac{C}{2}t^2}} + \frac{1}{C}$$

Indetermined Form,

$$\lim_{t\to\infty}\frac{t}{e^{\frac{C}{2}t^2}} = \frac{1}{C}\lim_{t\to\infty}\frac{1}{te^{\frac{C}{2}t^2}} \quad \text{(L'Hospital Rule)}$$

Use Squeeze Theorem

$$0 \le \frac{1}{te^{\frac{C}{2}t^2}} \le \frac{1}{e^t}$$

$$\lim_{t\to\infty}\frac{1}{e^t} = 0$$

Therefore

$$\sqrt{\frac{C}{2\pi}} \cdot \frac{1}{C}\lim_{t\to\infty}\frac{t}{e^{-\frac{C}{2}t^2}} = 0$$

Finally

$$\sigma^2 = \frac{1}{C}$$

$$C = \frac{1}{\sigma^2}$$

Finally, The Gaussian Distribution

$$p(x) = \sqrt{\frac{C}{2\pi}} e^{-C\frac{(x-\mu)^2}{2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$-END-$$