Laplace Transform

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First created: 2013. Last update: 2017-Feb-01

1 Theory

1.1 Definition

Time domain $t \longrightarrow s = \sigma + j\omega$ **C-Frequency domain**

$$F(s) = \mathcal{L}\{f\,(t)\} = \int_0^\infty e^{-st} f(t) dt$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} e^{st} F(s) dt$$

General one-sided L-Transform

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} u(t) f(t) dt$$

where unit step function (distribution selector)

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

The $\mathcal{L}$ is a limit

$$F(s) = \mathcal{L}\{f(t)\} = \lim_{N \to \infty} \int_0^N f\,(t) e^{-st} dt$$

1.2 Laplace Transform of Standard Function

Unity $\mathcal{L}\{1\}$

$$\mathcal{L}\{1\} = \int_0^\infty 1 \cdot e^{-st} dt = \frac{-1}{s} \int_0^\infty de^{-st} = \frac{-1}{s} \left[e^{-st} \right]_0^\infty = \frac{1}{s}$$

Linear Shift $\mathcal{L}\{e^{at}\}$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-(s-a)t} dt = \frac{-1}{s-a} \left[e^{-(s-a)t} \right]_0^\infty = \frac{1}{s-a}$$

Polynomial $\mathcal{L}\{t^n\}$

$$\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st} dt = \frac{-1}{s}\int_0^\infty t^n de^{-st} = \frac{-1}{s}\left[t^n e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} dt^n$$

Recall, the limit $\lim\limits_{N\to\infty}\frac{t^N}{e^N} = \lim\limits_{N\to\infty}\frac{Nt^{N-1}}{e^N} = ... = 0$ **(L'Hospital Rule)**

$$= \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt$$

Then,

$$= \frac{n}{s}\cdot\frac{n-1}{s}\int_0^\infty t^{n-2}e^{-st}dt = ... = \frac{n!}{s^n}\mathcal{L}\{1\} = \frac{n!}{s^{n+1}}$$

Expoential $\mathcal{L}\{t^a\}$

$$\mathcal{L}\{t^a\} = \int_0^\infty t^a e^{-st}dt$$

Let $u = st \longrightarrow \begin{cases} t = \frac{u}{s} \\ dt = \frac{du}{s} \end{cases}$

$$= \int_0^\infty \left(\frac{u}{s}\right)^a e^{-u}\frac{du}{s} = \frac{1}{s^{a+1}}\int_0^\infty u^a e^{-u}du = \frac{\Gamma(a+1)}{s^{a+1}}$$

Gamma Function $\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du$

Property of $\Gamma(x)$

$$\Gamma(a+1) = \int_0^\infty u^a e^{-u}du = -\int_0^\infty u^a de^{-u}$$

$$-\left[\frac{u^a}{e^u}\right] + a\int_0^\infty u^{a-1}e^{-u}du = a\Gamma(a)$$

$$\Gamma(1) = \int_0^\infty u^0 e^{-u}du = 1$$

Generalize, for $n \in \mathbb{N}^+$

$$\Gamma(n+1) = n\Gamma(n) = ... = n(n-1)(a-2)...\Gamma(1) = n!$$

Special case $\Gamma(\frac{1}{2})$

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-u}u^{-\frac{1}{2}}du = \int_0^\infty e^{-u}\frac{du}{\sqrt{u}}$$

Recall , $\frac{du}{\sqrt{u}} = 2d\sqrt{u}$

$$= 2\int_0^\infty e^{-u}d\sqrt{u}$$

Let $\sqrt{u} = x \rightarrow \begin{cases} d\sqrt{u} = dx \\ u = x^2 \end{cases}$

$$= 2\int_0^{\infty} e^{-x^2} dx$$

Tricks on integration of exp$(-x^2)$

$$= 2\left\{\int_0^{\infty} e^{-x^2} dx\right\}^{\frac{2}{2}} = 2\left\{\int_0^{\infty} e^{-x^2} dx \cdot \int_0^{\infty} e^{-y^2} dy\right\}^{\frac{1}{2}} = 2\left\{\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dxdy\right\}^{\frac{1}{2}}$$

Coordinate Transform

Rectangular $(x\,,\,y) \longrightarrow$ **Polar** $(r\,,\,\theta)$

$(r\,,\,\theta) \begin{cases} dxdy = rdrd\theta \\ x^2 + y^2 = r^2 \end{cases} \quad x, y\, \in [0\, + \infty] \longrightarrow \begin{cases} r \in [0, \infty] \\ \theta \in [0, \frac{\pi}{2}] \end{cases}$ **First Quardant**

$$= 2\left\{\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} rdrd\theta\right\}^{\frac{1}{2}}$$

$rdr = \frac{1}{2}dr^2$

$$= 2\left\{\frac{1}{2}\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} dr^2 d\theta\right\}^{\frac{1}{2}} = 2\sqrt{\frac{1}{2}\int_0^{\pi/2} \left[-e^{-r^2}\right]_0^{\infty} d\theta} = 2\sqrt{\frac{1}{2}\int_0^{\pi/2} d\theta} = 2\sqrt{\frac{1}{2}\cdot\frac{\pi}{2}}$$

$$= \sqrt{\pi}$$

SinCos $\mathcal{L}\{e^{j\omega t}\}$

Find $\mathcal{L}\{$**sin**$\omega t\}$ **by integration-by-parts**

$$\mathcal{L}\{\sin\omega t\} = \int_0^{\infty} \sin(\omega t)e^{-st}dt = \frac{-1}{s}\int_0^{\infty} \sin(\omega t)de^{-st} = \left[\frac{-1}{s}\sin(\omega t)e^{-st}\right]_0^{\infty} + \frac{a}{s}\int_0^{\infty} \cos(\omega t)e^{-st}dt$$

Find $\mathcal{L}\{e^{j\omega t}\} = \mathcal{L}\{$**cos**$\omega t + j$**sin**$\omega t\}$

$$\mathcal{L}\{e^{j\omega t}\} = \frac{1}{s - j\omega} \text{ (By Shifting Property)}$$

$$= \frac{1}{s - j\omega}\cdot\frac{s + j\omega}{s + j\omega} = \frac{s}{s^2 + \omega^2} + j\frac{\omega}{s^2 + \omega^2}$$

$$\therefore \mathcal{L}\{\cos\omega t\} = \frac{s}{s^2 + \omega^2} \quad \mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2 + \omega^2}$$

Find $\mathcal{L}\{e^{at}$**sin**$bt\}$

Hyperbolic SinCos $\mathcal{L}\{\text{sinh}\omega t\}$ $\mathcal{L}\{\text{cosh}\omega t\}$

$$\mathcal{L}\{\text{sinh}\omega t\} = \mathcal{L}\{\sin(j\omega t)\} = \int_0^\infty \frac{e^{\omega t} - e^{-\omega t}}{2} \cdot e^{-st} dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-\omega)t} - e^{-(s-\omega)t} dt = \frac{1}{2}\left(\frac{1}{s+\omega} - \frac{1}{s-\omega}\right) = \frac{\omega}{s^2 - \omega^2}$$

$$\mathcal{L}\{\text{cosh}\omega t\} = \mathcal{L}\{\cos(j\omega t)\} = \int_0^\infty \frac{e^{\omega t} + e^{-\omega t}}{2} \cdot e^{-st} dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-\omega)t} + e^{-(s-\omega)t} dt = \frac{1}{2}\left(\frac{1}{s+\omega} + \frac{1}{s-\omega}\right) = \frac{s}{s^2 - \omega^2}$$

1.3 Laplace Transform of Special Function

Heaviside Unit Step Function $\mathcal{L}\{u(t)\}$

$$\mathcal{L}\{u(t)\} = \int_0^\infty u(t) e^{-st} dt = \int_0^\infty e^{-st} dt = \frac{1}{s}$$

where $u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t \neq 0 \end{cases}$

the effect of unit step function is change the range of function from $\Re_0$ <u>*on to*</u> $[0\,,\,\infty]$
Definition of $f(t)$ that $t < 0$ is ignored / un-important.

$$\mathcal{L}\{u(t) f_1(t)\} = \mathcal{L}\{f_2(t)\}$$

$f_1(t) : D_1 \longmapsto R_1$
$f_2 : D_1 \longmapsto [0\,,\,\infty]$

General One Sided Laplace Transfrom for any function

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t) u(t) e^{-st} dt$$

General One Sided Laplace Transform for any function start at t_0

$$\mathcal{L}\{u(t_0 - t) f(t)\} = \int_0^\infty f(t) u(t_0 - t) e^{-st} dt$$

Ramp Function $\mathcal{L}\{r(t)\}$

$$r(t) = \begin{cases} t \cdot u(t) = t & t > 0 \\ 0 & t = 0 \end{cases}$$

$$\mathcal{L}\{r(t)\} = \int_0^\infty t e^{-st} dt = \mathcal{L}\{t^1\} = \frac{1}{s^2}$$

Dirac Delta Function $\mathcal{L}\{\delta(t)\}$

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$\int_{\mathbb{R}} \delta(t)dt = \int_{-\infty}^{\infty} \delta(t) = 1$$

Sign Function $\mathcal{L}\{\text{sgn}t\}$

$$\text{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

$$\mathcal{L}\{\text{sgn}t\} = \int_{-\infty}^{+\infty} \text{sgnt}\, e^{-st}dt = \int_0^{\infty} e^{-st}dt - \int_{-\infty}^{0} e^{-st}dt = 2\int_0^{\infty} e^{-st}dt = \frac{2}{s}$$

1.4 Properties of $\mathcal{L}$

$\mathcal{L}$ is a linear operator

$$\mathcal{L}(a \cdot f(t) + b \cdot g(t)) = a \cdot \mathcal{L}\{f(t)\} + b \cdot \mathcal{L}\{g(t)\}$$

$$\mathcal{L}(a \cdot f(t) + b \cdot g(t)) = \int_0^{\infty} [a \cdot f(t) + b \cdot g(t)]\, e^{-st}dt$$

$$= a\int_0^{\infty} f(t)e^{-st}dt + b\int_0^{\infty} g(t)e^{-st}dt = a \cdot \mathcal{L}\{f(t)\} + b \cdot \mathcal{L}\{g(t)\}$$

Expoential Linear Shift $\mathcal{L}\{e^{at}\} = \mathcal{L}\{e^{-(s-a)t}\}$

$$\mathcal{L}\{e^{at}\} = \int_0^{\infty} e^{at}e^{-st}dt = \int_0^{\infty} e^{-(s-a)t}dt = \frac{1}{s-a}$$

Unity Linear Shift $\mathcal{L}\{u(t-t_0)f(t)\}$

$$\mathcal{L}\{u(t_0-t)f(t)\} = \int_0^{\infty} u(t_0-t)f(t)e^{-st}dt = \int_{t_0}^{\infty} f(t)e^{-st}dt$$

Scaling $\mathcal{L}\{f(at)\} = \frac{1}{|a|}\mathcal{L}\{f(t)\}_{s'=\frac{s}{a}} = \frac{1}{a}F\left(\frac{s}{a}\right)$

$$\mathcal{L}\{f(at)\} = \int_0^{\infty} f(at)e^{-st}dt$$

Let $u = at \rightarrow \begin{cases} t = \dfrac{u}{a} \\ dt = \dfrac{du}{a} \end{cases}$

$$= \int_0^{\infty} f(u)e^{-\frac{su}{a}}\frac{du}{a} = \frac{1}{a}\int_0^{\infty} f(u)e^{-\frac{s}{a}u}du = \frac{1}{a}\mathcal{L}\left\{f(t)\right\}_{s'=\frac{s}{a}}$$

Differentiation $\mathcal{L}\{\frac{df(t)}{dt}\}$

$$\mathcal{L}\{\frac{df(t)}{dt}\} = \int_0^\infty \frac{df(t)}{dt} e^{-st} dt = \int_0^\infty e^{-st} df(t)$$

$$= \left[e^{-st} f(t)\right]_0^\infty + \int_0^\infty f(t) de^{-st} = f(0) + s\mathcal{L}\{f(t)\}$$

Generalize,

$$\mathcal{L}\{\frac{d^n f(t)}{dt^n}\} = \left[\frac{d^n f(t)}{dt^n}\right]_{t=0} + s\mathcal{L}\left\{\frac{d^{n-1} f(t)}{dt^{n-1}}\right\} = ... = f^{(n)}(t)_{t=0} + sf^{(n-1)}(t)_{t=0} + s^{n-1}$$

Integration $\mathcal{L}\{\int_0^t f(x)dx\}$

$$\mathcal{L}\{\int_0^t f(x)dx\} = \int_0^\infty \left\{\int_0^t f(x)dx\right\} e^{-st} dt = \frac{-1}{s} \int_0^\infty \left\{\int_0^t f(x)dx\right\} de^{-st}$$

$$= \frac{-1}{s} \int_0^t f(x)dx \cdot e^{-st}\Big|_{t=0}^{t=\infty} + \frac{1}{s} \int_0^\infty e^{-st} \mathrm{d}\left(\int_0^t f(x)dx\right)$$

$$= \frac{1}{s} \int_0^\infty e^{-st} \frac{\mathrm{d}}{\mathrm{dt}} \left(\int_0^t f(x)dx\right) \mathrm{dt} = \frac{1}{s} \int_0^\infty e^{-st} f(x)dt = \frac{1}{s} F(s)$$

Generalize,

$$\mathcal{L}\{\int_0^t ... \int_0^t f(x)dx^n\} = \frac{F(s)}{s^n}$$

$t-$**multiplying** $\mathcal{L}\{tf(t)\}$

Convolution

2 Application

2.1 Linear System and Convolution

System Relation in time domain is a convolution

$$y(t) = x(t) * g(t) = \int_0^\infty x(t-\tau) g(\tau) d\tau = \int_0^\infty x(\tau) g(t-\tau) d\tau$$

By L-transform,

$$Y(s) = \int_0^\infty e^{-st} y(t) dt$$

$$= \int_0^\infty e^{-st} \left(\int_0^\infty x(t-\tau) g(\tau) d\tau\right) dt$$

$$\text{Let } t' = t - \tau \implies \begin{cases} t = t' + \tau \\ dt = d\tau \end{cases}$$

$$= \int_0^\infty e^{-s(t'+\tau)} \left(\int_0^\infty r(t')g(\tau)d\tau \right) dt'$$

$$= \int_0^\infty e^{-st'} x(t')dt' \int_0^\infty e^{-s\tau} g(\tau)d\tau$$

$$= X(s)G(s)$$

In frequency domain,

$$Y(s) = X(s)G(s)$$

$\therefore$via $\mathcal{L}$, complicated calculus problem become a simple algebra problem

3 Inverse Laplace

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds$$

This integral is difficult to compute.
Computation of inverse Laplace Transfrom via D-contour using complex calculus

$$\int_{b-j\infty}^{b+j\infty} F(s)e^{st}ds + \int_{C_R} F(s)e^{st}ds = \oint_D F(s)e^{st}ds$$

Since

$$\int_{C_R} F(s)e^{st}ds = 0 \quad \text{for } R \to \infty$$

Thus

$$\oint_D F(s)e^{st}ds = \sum Res\left\{F, z_k\right\}$$

$\therefore$

$$f(t) = \frac{1}{2\pi j} \lim_{R\to\infty} \int_{b-jR}^{b+jR} F(s)e^{st}ds = \oint_D F(s)e^{st}ds - \int_{b-j\infty}^{b+j\infty} F(s)e^{st}ds$$

Therefore the inverse Laplace Transform can be computed using residue as

$$f(t) = \oint_D F(s)e^{st}ds = \sum Res\left\{F, z_k\right\}$$

Find Inverse via Linearity
If a complicated function can be broken down into components:

$$F(s) = F_1(s) + F_2(s) + ... + F_n(s)$$

Then the inverse Laplace transform can be computed by summing the inverse of all components together

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}\{F_1(s) + F_2(s) + ... + F_n(s)\} = \sum \mathcal{L}^{-1}\{F_i(s)\}$$

When the component is *simple*, a table-look up approach can be used to speed up the computation.

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