Orthogonal Functions in Fourier Analysis

Ang Man Shun

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Summary

$$\left\langle \cos nx, \cos mx \right\rangle = \begin{cases} \left\langle \cos nx, 1 \right\rangle_{m=0} = 0 \\ 0 & m \neq n \\ \pi & m = n \neq 0 \\ 2\pi & m = n = 0 \end{cases} \qquad \left\langle e^{jn\omega t}, e^{jm\omega t} \right\rangle = \begin{cases} \left\langle e^{jn\omega t}, 1 \right\rangle_{m=0} = 0 \\ 0 & m \neq n \\ 2\pi & m = n \end{cases}$$

$$\left\langle \sin nx, \sin mx \right\rangle = \begin{cases} \left\langle \sin nx, 1 \right\rangle_{\sin mx=1} = 0 \\ 0 & m \neq n \\ \pi & m = n \end{cases} \qquad \left\langle \cos nx, \sin mx \right\rangle = 0 \qquad \forall m, n$$

Review of vector

- Inner product of $\bar{a}$, $\bar{b}$: $\bar{a} \cdot \bar{b} = \sum_{k=1}^{k=N} a_k b_k$
- Orthogonal : Vector $\bar{a}$, $\bar{b}$ are orthogonal to each other $\iff$ $\bar{a} \cdot \bar{b} = \sum a_k b_k = 0$

Inner Product for function The *inner product* of f , g w.r.t. weighting r is defined as

$$\left\langle f , g \right\rangle_r = \int f(t)g(t)r(t)dt$$

*Ususally, weighting function $r(t)$ is unity function , $r(t) = 1(t) = 1$

If f , g is complex function

$$\left\langle f , g \right\rangle_r = \int f(t)\overline{g(t)}r(t)dt$$

Orthogonal Two function f , g is *orthogonal* to each other if their inner product is zero

$$f\,g \text{ are orthogonal} \iff \left\langle f , g \right\rangle_1 = 0$$

Orthogonal Pairs

$$\Big\langle \cos nx\,,\,1 \Big\rangle \qquad \Big\langle \sin nx\,,\,1 \Big\rangle \qquad \Big\langle \cos nx\,,\,1 \Big\rangle \qquad \Big\langle \sin nx\,,\,1 \Big\rangle$$

$$\Big\langle \cos nx\,,\,\sin mx \Big\rangle \qquad \Big\langle e^{jn\omega t}\,,\,e^{jm\omega t} \Big\rangle \qquad \Big\langle e^{jn\omega t}\,,\,1 \Big\rangle$$

$n, m \in \mathbb{Z}$
$\cos nx$ and DC term (which is the 1) are orthogonal

$$\Big\langle \cos nx\,,1 \Big\rangle = \int_0^{2\pi} \cos nx dx = \frac{1}{n}\sin nx|_0^{2\pi} = 0$$

$\sin nx$ and DC term (which is the 1) are orthognal

$$\Big\langle \sin nx\,,1 \Big\rangle = \int_0^{2\pi} \sin nx dx = \frac{-1}{n}\cos nx|_0^{2\pi} = 0$$

$\cos nx$ and $\cos mx$ are orthogonal

$$\Big\langle \cos nx\,,\,\cos mx \Big\rangle = \int_0^{2\pi} \cos nx \cos mx dx = \frac{1}{2}\int_0^{2\pi} \left[\cos\left(n-m\right)x + \cos\left(n+m\right)x\right] dx$$

$$= \frac{1}{2}\int_0^{2\pi} \cos\left(n-m\right)x dx + \frac{1}{2}\int_0^{2\pi} \cos\left(n+m\right)x dx = \frac{1}{2}\Big\langle \cos(n-m)x, 1 \Big\rangle + \frac{1}{2}\Big\langle \cos\left(n+m\right)x, 1 \Big\rangle = 0$$

$\sin nx$ and $\sin mx$ are orthogonal

$$\Big\langle \sin nx\,,\sin mx \Big\rangle = \int_0^{2\pi} \sin nx \sin mx dx = \frac{1}{2}\int_0^{2\pi} \left[\cos\left(n-m\right)x - \cos\left(n+m\right)x\right] dx$$

$$= \frac{1}{2}\int_0^{2\pi} \cos\left(n-m\right)x dx - \frac{1}{2}\int_0^{2\pi} \cos\left(n+m\right)x dx = \frac{1}{2}\Big\langle \cos(n-m)x, 1 \Big\rangle - \frac{1}{2}\Big\langle \cos\left(n+m\right)x, 1 \Big\rangle = 0$$

$\cos nx$ and $\sin mx$ are orthogonal to each other

$$\Big\langle \cos nx\,,\,\sin mx \Big\rangle = \int_0^{2\pi} \cos nx \sin mx dx = \frac{1}{2}\int_0^{2\pi} \left(\sin\left(n+m\right)x - \sin\left(n-m\right)x\right) dx$$

$$= \frac{1}{2}\int_0^{2\pi} \sin\left(n+m\right)x dx - \frac{1}{2}\int_0^{2\pi} \sin\left(n-m\right)x dx = \frac{1}{2}\underbrace{\Big\langle \sin(n+m)x\,,1 \Big\rangle}_{0} - \frac{1}{2}\underbrace{\Big\langle \sin(n-m)x\,,1 \Big\rangle}_{0} = 0$$

$e^{jn\omega t}$ and $e^{jm\omega t}$are orthogonal to each other

$$\Big\langle e^{jn\omega t}\,,e^{jm\omega t} \Big\rangle = \int_0^{2\pi} e^{jn\omega t}e^{-jm\omega t} dt = \int_0^{2\pi} e^{j(n-m)\omega t} dt = \left.\frac{e^{j(n-m)\omega t}}{j(n-m)\omega t}\right|_{t=0}^{t=2\pi} = 0$$

$e^{jn\omega t}$ and DC term (which is the 1) are orthogonal

$$\Big\langle e^{jn\omega t}\,,1 \Big\rangle = \int_0^{2\pi} e^{jn\omega t} dt = \left.\frac{e^{jn\omega t}}{jn\omega t}\right|_{t=0}^{t=2\pi} = 0$$

When $n = m$

$$\Big\langle \cos nx \, , \, \cos mx \Big\rangle \qquad \Big\langle \sin nx \, , \, \sin mx \Big\rangle \qquad \Big\langle \cos nx \, , \, \sin mx \Big\rangle \qquad \Big\langle e^{jn\omega t} \, , \, e^{jm\omega t} \Big\rangle$$

$$\Big\langle \cos nx \, , \, \cos mx \Big\rangle_{n=m} = \int_0^{2\pi} \cos^2 nx dx = \int_0^{2\pi} \frac{1 + \cos 2nx}{2} dx = \frac{1}{2} \int_0^{2\pi} dx + \frac{1}{2} \int_0^{2\pi} \cos 2nx dx$$

$$= \frac{1}{2} \int_0^{2\pi} dx + \frac{1}{2} \Big\langle \cos 2nx \, , \, 1 \Big\rangle = \pi$$

$$\Big\langle \sin nx \, , \, \sin mx \Big\rangle_{n=m} = \int_0^{2\pi} \sin^2 nx dx = \int_0^{2\pi} \frac{1 - \cos 2nx}{2} dx = \frac{1}{2} \int_0^{2\pi} dx + \frac{1}{2} \int_0^{2\pi} \cos 2nx dx$$

$$= \frac{1}{2} \int_0^{2\pi} dx + \frac{1}{2} \Big\langle \cos 2nx \, , \, 1 \Big\rangle = \pi$$

$$\Big\langle \cos nx \, , \, \sin mx \Big\rangle_{n=m} = \int_0^{2\pi} \sin nx \cos nx dx = \frac{1}{2} \int_0^{2\pi} \sin 2nx dx = \frac{1}{2} \Big\langle \sin 2nx, 1 \Big\rangle = 0$$

$$\Big\langle e^{jn\omega t} \, , \, e^{jm\omega t} \Big\rangle_{n=m} = \int_0^{2\pi} e^{jn\omega t} e^{-jm\omega t} dt = \int {}^{2\pi}_{0} dt = 2\pi$$

Summary

$$\Big\langle \cos nx, \cos mx \Big\rangle = \begin{cases} \pi \delta_{m,n} & m, n \neq 0 \\ 2\pi & m = n = 0 \end{cases} \qquad \Big\langle e^{jn\omega t}, e^{jm\omega t} \Big\rangle = 2\pi \delta_{m,n}$$

$$\Big\langle \sin nx, \sin mx \Big\rangle = \pi \delta_{m,n} \qquad \Big\langle \cos nx, \sin mx \Big\rangle = 0 \qquad \forall m, n$$

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