Fourier Coefficient

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1 Orthogonality

$$\left\langle \cos nx, \cos mx \right\rangle = \pi\delta_{mn} \qquad \left\langle e^{jn\omega t}, e^{jm\omega t} \right\rangle = 2\pi\delta_{m,n}$$

$$\left\langle \sin nx, \sin mx \right\rangle = \pi\delta_{mn} \qquad \left\langle \cos nx, \sin mx \right\rangle = 0 \qquad \forall m, n$$

2 Real Fourier Series

Any periodic function $f(t)$ can be represented by sum of trigonometric series / Any periodic function $f(t)$ can be broken down into it's frequency components

$$f(x) = \sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx$$

2.1 Extraction of a_k

To extract the Fourier Coefficient a_k , consider the inner product of the function

$$\langle f(t)\,,\ \cos kx \rangle = \int_0^{2\pi} f(t) \cos kx dx$$

Plug in the Fourier Expansion

$$\langle f(t)\,,\ \cos kx \rangle = \int_0^{2\pi} \left[\sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx \right] \cos kx dx$$

$$= \sum_{n=0}^{\infty} a_n \int_0^{2\pi} \cos nx \cos kx dx + b_n \int_0^{2\pi} \sin nx \cos kx dx$$

By $\left\langle \cos nx, \cos mx \right\rangle = \pi\delta_{mn}$ $\quad \left\langle \cos nx, \sin mx \right\rangle = 0$, all the terms except $n = k$ goes to zero

$$= a_k \pi$$

$\therefore$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx dx$$

2.2 Extraction of b_k

To extract Fourier Coefficient b_k , consider the inner product of the function

$$\langle f(x)\,,\,\sin kx\rangle = \int_0^{2\pi} f(t)\sin kxdx$$

Plug in the Fourier Expansion

$$\langle f(x)\,,\,\sin kx\rangle = \int_0^{2\pi}\left[\sum_{n=0}^{\infty} a_n\cos nx + b_n\sin nx\right]\sin kxdx$$

$$= \sum_{n=0}^{\infty} a_n\int_0^{2\pi}\cos nx\sin kxdx + b_n\int_0^{2\pi}\sin nx\sin kxdx$$

By $\Big\langle \sin nx, \sin mx\Big\rangle = \pi\delta_{mn}$ $\quad \Big\langle \cos nx, \sin mx\Big\rangle = 0$, all the terms except $n = k$ go to zero

$$= b_k\pi$$

$\therefore$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kxdx$$

2.3 Extraction of a_0

To extract a_0, consider the inncer product

$$\langle f(x)\,,\,1\,\rangle = \int_0^{2\pi} f(x)dx$$

Plug in the Fourier Expansion

$$\langle f(x)\,,\,1\rangle = \sum_{n=0}^{\infty} a_n \underbrace{\int_0^{2\pi}\cos nxdx}_{\text{All zero except } n=0} + b_n\underbrace{\int_0^{2\pi}\sin nxdx}_{\text{All zero}} = a_0\int_0^{2\pi}dx = 2\pi a_0$$

$\therefore$

$$a_0' = 2a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx \quad \Rightarrow \text{First terms is } \frac{a_0'}{2}$$

2.4 Summary

$$f(x) = \frac{a_0'}{2} + \textstyle\sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx \quad - \quad a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kxdx$$

$$a_0' = \frac{1}{\pi}\textstyle\int_0^{2\pi} f(x)dx \qquad b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kxdx$$

3 Complex Fourier Series

Apply Euler's Formula in Real Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{e^{jnx} + e^{-jnx}}{2}\right) + b_n \left(\frac{e^{jnx} - e^{-jnx}}{2j}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n + \frac{b_n}{j}}{2} e^{jnx} + \frac{a_n - \frac{b_n}{j}}{2} e^{-jnx} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{jnx} + \frac{a_n + jb_n}{2} e^{-jnx}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{jnx} + \sum_{n=1}^{\infty} \frac{a_n + jb_n}{2} e^{-jnx} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{jnx} + \sum_{n=-\infty}^{-1} \frac{a_n + jb_n}{2} e^{jnx} = \sum_{n=-\infty}^{\infty} c_n e^{jnx}$$

$\therefore$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{jnx}$$

Where

$$c_0 = \frac{a_0}{2} \qquad c_n = \frac{a_n - jb_n}{2} \qquad c_{-n} = \frac{a_n + jb_n}{2} = \bar{c}_n$$

3.1 Extraction of c_k

To extract c_k , consider inner product

$$\langle f(x), e^{jkx} \rangle = \int_0^{2\pi} f(x) e^{-jkx} dx$$

Plug in the Complex Fourier Series expansion

$$\langle f(x), e^{jkx} \rangle = \int_0^{2\pi} \left[\sum_{n=-\infty}^{\infty} c_n e^{jnx}\right] e^{-jkx} dx = \sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} e^{jnx} e^{-jkx} dx = \sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} e^{j(n-k)x} dx$$

$$= \sum_{\substack{n=-\infty \\ n \neq k}}^{\infty} c_n \int_0^{2\pi} e^{j(n-k)x} dx + c_k \int_0^{2\pi} dx$$

By orthogonality ,$\langle e^{jnx}, e^{jkx} \rangle = 2\pi\delta_{m,n}$

$$\sum_{\substack{n=-\infty \\ n \neq k}}^{\infty} c_n \int_0^{2\pi} e^{j(n-k)x} dx \quad \text{All equal zero}$$

$$\langle f(x), e^{jkx} \rangle = c_k 2\pi$$

$\therefore$

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-jkx} dx$$

4 Summary

Any periodic function $f(t)$ can be represented by sum of trigonometric series / Any periodic function f(t) can be broken down into it's frequency components

	Real Fourier Series	Complex Fourier Series
Form	$f(t) = \frac{a'_0}{2} + \sum_{n=1}^{N} a_n \cos nx + b_n \sin nx$	$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnt}$
Coefficients	$a'_0 = 2a_0 = \frac{1}{\pi} \int_0^{2\pi} f(t)\, dt$ $a_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos kt\, dt$ $b_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin kt\, dt$	$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-jkt} dt$

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