# Fourier Transform Paris

Andersen Ang

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## 1 Fourier Transform Pairs

### 1.1 $f(t) = 1 \qquad a < x < b \qquad 1 \longleftrightarrow \dfrac{e^{-j\omega a} - e^{-j\omega b}}{j\omega}$

*Proof.*

$$\mathcal{F}\{1\} = \int_{-\infty}^{\infty} e^{-j\omega t} dt = \int_a^b e^{-j\omega t} dt = \frac{1}{-j\omega} \left[e^{-j\omega t}\right]_a^b = \frac{e^{-j\omega b} - e^{-j\omega a}}{-j\omega} = \frac{e^{-j\omega a} - e^{-j\omega b}}{j\omega}$$

□

### 1.2 $f(t) = 1 \qquad -a < t < a \qquad 1 \longleftrightarrow \dfrac{2\sin a\omega}{\omega} = 2a\textbf{sinc}a\omega$

*Proof.*

$$\mathcal{F}\{1\} = \int_{-\infty}^{\infty} e^{-j\omega t} dt = \int_{-a}^{a} e^{-j\omega t} dt = \frac{1}{-j\omega} \left[e^{-j\omega t}\right]_{-a}^{a} = \frac{e^{-j\omega a} - e^{j\omega a}}{-j\omega} = \frac{e^{j\omega a} - e^{-j\omega a}}{j\omega}$$

$$= \frac{2}{\omega} \frac{e^{j\omega a} - e^{-j\omega a}}{2j} = \frac{2\sin a\omega}{\omega} = 2a\frac{\sin a\omega}{a\omega} = 2a\text{sinc}a\omega$$

□

### 1.3 $e^{-ct} \qquad a < t < b$

*Proof.*

$$\mathcal{F}\left\{e^{-ct}\right\} = \int_{-\infty}^{\infty} e^{-ct} e^{-j\omega t} dt = \int_a^b e^{-(j\omega + c)t} dt = \frac{1}{-(j\omega + c)} \left[e^{-(j\omega+c)t}\right]_a^b$$

$$= \frac{e^{-(j\omega+c)b} - e^{-(j\omega+c)a}}{-j\omega} = \frac{e^{-(j\omega+c)a} - e^{-(j\omega+c)b}}{j\omega}$$

□

### 1.4 $e^{-ct} \qquad t > 0 \quad Re\,(c) > 0$

*Proof.*

$$\mathcal{F}\left\{e^{-ct}\right\} = \int_{-\infty}^{\infty} e^{-ct} e^{-j\omega t} dt = \int_0^{\infty} e^{-(j\omega+c)t} dt = \frac{1}{-(j\omega + c)} \underbrace{\left[e^{-(j\omega+c)t}\right]_0^{\infty}}_{-1} = \frac{1}{j\omega + c}$$

□

## 1.5 $e^{-c|t|}$ $\quad Re(c) > 0$

*Proof.*

$$\mathcal{F}\left\{e^{-c|t|}\right\} = \int_{-\infty}^{\infty} e^{-c|t|} e^{-j\omega t} dt = \int_{-\infty}^{0} e^{ct} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-ct} e^{-j\omega t} dt$$

Take $t = -\tau$ , $dt = -d\tau$, $t = 0 \to \tau = 0$ , $t = -\infty \to \tau = +\infty$

$$= -\int_{+\infty}^{0} e^{-c\tau} e^{-j\omega t} d\tau + \int_{0}^{\infty} e^{-ct} e^{-j\omega t} dt = -\int_{+\infty}^{0} e^{-(j\omega+c)t} dt + \int_{0}^{\infty} e^{-(j\omega+c)t} dt$$

$$= \int_{0}^{\infty} e^{-(j\omega+c)t} dt + \int_{0}^{\infty} e^{-(j\omega+c)t} dt = 2\int_{0}^{\infty} e^{-(j\omega+c)t} dt$$

$$= 2\frac{\left[e^{-(j\omega+c)t}\right]_0^{\infty}}{-(j\omega+c)} = \frac{2}{j\omega+c}$$

Rationalize

$$\frac{2}{j\omega+c} \cdot \frac{j\omega-c}{j\omega-c} = \frac{2j\omega-2c}{\omega^2+c^2} ?????$$

□

$\delta(t)$

$$\mathcal{F}\left\{\delta(t)\right\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t) \underbrace{e^0}_{1} dt = 1$$

$\delta(t)$

$$\delta(t) \longleftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} dt$$

$\cos\omega t$

$$\mathcal{F}\left\{\cos at\right\} = \int_{-\infty}^{\infty} \cos at\, e^{-j\omega t} dt = \int_{-\infty}^{\infty} \cos at\left(\cos\omega t - j\sin\omega t\right) dt$$

$$= \int_{-\infty}^{\infty} \cos\omega t \cos at\, dt - j\underbrace{\int_{-\infty}^{\infty} \cos at \sin\omega t dt}_{Zero} = \begin{cases} 0 & \omega \neq \pm a \\ \int_{-\infty}^{\infty} \cos^2 at\, dt & \omega = \pm a \end{cases}$$

## 1.6 Rectangular Pulse $\sqcap(t)$

$$\sqcap(t) = \begin{cases} A & -\frac{\tau}{2} \leq t \leq \frac{\tau}{2} \\ 0 & else \end{cases}$$

$$\mathcal{F}\left\{\sqcap(t)\right\} = \int_{-\infty}^{\infty} \sqcap(t) e^{-j\omega t} dt = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A e^{-j\omega t} dt = \frac{A}{-j\omega}\left[e^{-j\omega t}\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \frac{-2A}{\omega}\left[\frac{e^{-j\omega\frac{\tau}{2}} - e^{j\omega\frac{\tau}{2}}}{2j}\right] = \frac{2A}{\omega}\left[\sin\omega\frac{\tau}{2}\right]$$

$$\mathcal{F}\left\{\sqcap(t)\right\} = A\tau\frac{\sin\omega\frac{\tau}{2}}{\omega\frac{\tau}{2}} = A\tau \text{sinc}\omega\frac{\tau}{2}$$

## 1.7 Triangular Pulse $\wedge(t)$

Triangular Pulse has different form of representation

$$\wedge(t) = \begin{cases} t & -\frac{\tau}{2} \leq t \leq \frac{\tau}{2} \\ 0 & else \end{cases} \qquad or \quad \wedge(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & else \end{cases}$$

But when considering the Fourier Transform, treat Triangular Pulse as convolution of rectangular pulse

$$\wedge(t) = \sqcap(t) * \sqcap(t) = \int_{-\infty}^{\infty} \sqcap(\tau) \sqcap (t - \tau) d\tau$$

Thus, the Fourier Transform of $\wedge(t)$ is

$$\mathcal{F}\{\wedge(t)\} = \mathcal{F}\{\sqcap(t) * \sqcap(t)\} = \mathcal{F}\{\sqcap(t)\} \cdot \mathcal{F}\{\sqcap(t)\} = \mathcal{F}\{\sqcap(t)\}^2$$

$$\mathcal{F}\{\wedge(t)\} = \left(A\tau \text{sinc}\omega\frac{\tau}{2}\right)^2$$

## 1.8 Boxcar Pulse $Bc(t)$

$$Bc(t) = \sqcap(t - a) - \sqcap(t - b)$$

Fourier Transform of Boxcar pulse can be treated as an application of Fourier Transform property

$$\sqcap(t) \xleftrightarrow{\mathcal{F}} A\tau \text{sinc}\omega\frac{\tau}{2}$$

Thus

$$\sqcap(t - \tau) \xleftrightarrow{\mathcal{F}} e^{j\omega\tau}\left(A\tau \text{sinc}\omega\frac{\tau}{2}\right)$$

Therefore

$$\sqcap(t - a) - \sqcap(t - b) \xleftrightarrow{\mathcal{F}} e^{j\omega a}\left(A\tau \text{sinc}\omega\frac{\tau}{2}\right) - e^{j\omega b}\left(A\tau \text{sinc}\omega\frac{\tau}{2}\right) = A\tau \text{sinc}\omega\frac{\tau}{2}\left(e^{j\omega a} - e^{j\omega b}\right)$$

## 1.9 Dirac Delta

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1$$

*Proof.* □

$$\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega t}\Big|_{t=0} = 1$$