Properties of Fourier Transform - I

Ang M.S.

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Reference C.K. Alexander , M.N.O Sadiku *Fundamentals of Electric Circuits*

Summay

	Original Function	Transformed Function
1. Linear	$af_1(t) + bf_2(t)$	$aF_1(j\omega) + bF_2(j\omega)$
2. Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{j\omega}{a}\right)$
3. Time Shift	$f(t-t_0)$	$e^{-j\omega t_0}F(j\omega)$
4. Frequency Shift	$e^{at}f(t)$	$F(j(\omega - a))$
5. Reverse Time	$f(-t)$	$F(-j\omega)$
6. Time Differentiation	$\frac{df(t)}{dt}$ $\frac{d^n f(t)}{dt^n}$	$j\omega F(j\omega)$ $(j\omega)^n F(j\omega)$
7. Time Integral	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{1}{j\omega}F(j\omega)$
8. Frequency Differentiation t-multiplication	$tf(t)$ $t^n f(t)$	$j\frac{dF(j\omega)}{d\omega}$ $j^n\frac{d^n}{d\omega^n}F(j\omega)$
9. Frequency Integral $t -$ division	$\frac{f(t)}{t}$	$\int_{j\omega}^{\infty} F(j\omega)d\omega$

1 $\mathcal{F}$

1.1 Linear

$$ax_1(t) + bx_2(t) \longleftrightarrow aX_1(j\omega) + bX_2(j\omega)$$

Proof.

$$\mathcal{F}\left\{ax_1(t) + bx_2(t)\right\} = \int_{-\infty}^{\infty} \left(ax_1(t) + bx_2(t)\right) e^{-j\omega t}dt$$

$$= a\int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}dt + b\int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}dt = aX_1(j\omega) + bX_2(j\omega)$$

□

1.2 Scaling

$$x\left(at\right) \longleftrightarrow \frac{1}{a} X\left(\frac{j\omega}{a}\right)$$

Proof.

$$\mathcal{F}\left\{x\left(at\right)\right\} = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

Let $at = \tau$, $t = \frac{\tau}{a}$, $dt = \frac{1}{a} d\tau$, $t \in (-\infty, +\infty) \Rightarrow \tau \in (-\infty, +\infty)$

$$= \int_{-\infty}^{\infty} x\left(\tau\right) \exp\left(-j\frac{\omega}{a}\tau\right)\left(\frac{1}{a}d\tau\right) = \frac{1}{a}\int_{-\infty}^{\infty} x\left(\tau\right) \exp\left(-j\frac{\omega}{a}\tau\right) d\tau = \frac{1}{a} X\left(\frac{j\omega}{a}\right)$$

□

1.3 Time Shift / Modulation in Time

$$x\left(t - t_0\right) \longleftrightarrow e^{-j\omega t_0} X\left(j\omega\right)$$

Proof.

$$\mathcal{F}\left\{x\left(t - t_0\right)\right\} = \int_{-\infty}^{\infty} x\left(t - t_0\right) e^{-j\omega t} dt$$

Let $\tau = t - t_0$, $t = \tau + t_0$, $dt = d\tau$, $t \in (-\infty, \infty) \Longrightarrow \tau \in (-\infty, +\infty)$

$$= \int_{-\infty}^{\infty} x\left(\tau\right) e^{-j\omega(\tau + t_0)} d\tau = e^{-j\omega t_0} \int_{-\infty}^{\infty} x\left(\tau\right) e^{-j\omega\tau} d\tau = e^{-j\omega t_0} X\left(j\omega\right)$$

□

Remark. Compare to causal, unilateral$\mathcal{L}$, $\mathcal{F}$ is bilateral, Heaviside Unit Step Function is not required

1.4 Expoential Shift / Modulation in Frequency

$$e^{jat} x\left(t\right) \longleftrightarrow X\left(j(\omega - a)\right)$$

Proof.

$$\mathcal{F}\left\{e^{jat} x(t)\right\} = \int_{-\infty}^{\infty} e^{jat} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j(\omega - a)t} dt = X\left(j\left(\omega - a\right)\right)$$

□

1.5 Time Reverse

$$x(-t) \longleftrightarrow X\left(-j\omega\right)$$

Proof.

$$\mathcal{F}\left\{x(-t)\right\} = \int_{-\infty}^{\infty} x\left(-t\right) e^{-j\omega t} dt$$

Let $-t = \tau$, $dt = -d\tau$, $t = -\infty \Rightarrow \tau = \infty$, $t = \infty \Rightarrow \tau = -\infty$

$$= \int_{\infty}^{-\infty} x\left(\tau\right) e^{j\omega\tau} (-1) d\tau = \int_{-\infty}^{\infty} x\left(\tau\right) e^{-j(-\omega)\tau} d\tau = X(-j\omega)$$

□

1.6 Time-Differentiation

$$\frac{dx(t)}{dt} \longleftrightarrow j\omega X(j\omega)$$

Proof.

$$\mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-j\omega t} dx(t) = \underbrace{\left[\frac{e^{-j\omega t} x(t)}{-j\omega}\right]_{-\infty}^{\infty}}_{0\ x(\pm\infty)=0} - \int_{-\infty}^{\infty} x(t) de^{-j\omega t} = j\omega X(j\omega)$$

□

Remark 1. $\lim_{t\to\pm\infty} x(t) = 0$

Remark 2. Repeat the process

$$\mathcal{F}\left\{\frac{d}{dt}\left[\frac{dx(t)}{dt}\right]\right\} = j\omega\left[\mathcal{F}\left\{\frac{dx(t)}{dt}\right\}\right] = (j\omega)^2 X(j\omega)$$

$$\mathcal{F}\left\{\frac{d^n x(t)}{dt^n}\right\} = (j\omega)^n X(j\omega)$$

1.7 Time Integral

$$\int_{-\infty}^{t} f(\tau) d\tau \longleftrightarrow \frac{1}{j\omega} F(j\omega)$$

Proof. Consider $g(t) = \int_{-\infty}^{t} f(\tau) d\tau$, $\lim_{t\to\pm\infty} g(t) = 0$

$$\frac{dg(t)}{dt} = f(t)$$

$$\mathcal{F}\left\{\frac{dg(t)}{dt}\right\} = j\omega G(j\omega) \qquad \underset{\int}{\overset{\frac{d}{dt}}{\longleftrightarrow}} \qquad \mathcal{F}\{f(t)\} = F(j\omega)$$

$$\begin{cases} G(j\omega) = \mathcal{F}\{g(t)\} = \mathcal{F}\left\{\int_{-\infty}^{t} f(\tau) d\tau\right\} \\ j\omega G(j\omega) = F(j\omega) \end{cases}$$

$$\mathcal{F}\left\{\int_{-\infty}^{t} f(\tau) d\tau\right\} = \frac{1}{j\omega} F(j\omega)$$

□

1.8 Frequency Differentiation / t-multiplication

$$t^n x(t) \longleftrightarrow j^n \frac{d^n}{d\omega^n} X(j\omega)$$

Proof. Consider RHS for $n = 1$

$$\frac{dX(j\omega)}{d\omega} = \frac{d}{d\omega}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)\frac{de^{-j\omega t}}{d\omega}dt = \int_{-\infty}^{\infty} -jtx(t)e^{-j\omega t}dt = j\int_{-\infty}^{\infty} tx(t)e^{-j\omega t}dt$$

$$\therefore j\frac{dX\left(j\omega\right)}{d\omega} = \int_{-\infty}^{\infty} tx(t)e^{-j\omega t}dt$$

Repeat the process , or using $Mathematical\ Induction$,

$$\mathcal{F}\left\{t^n x(t)\right\} = j^n \frac{d^n}{d\omega^n}X(j\omega)$$

□

1.9 Frequency Integration / t-division

$$\frac{x(t)}{t} \longleftrightarrow \int_{j\omega}^{\infty} X\left(j\mu\right) dj\mu$$

Proof. Consider RHS :

$$\int_{j\omega}^{\infty}\left[X\left(j\mu\right)\right] dj\mu = \int_{j\omega}^{\infty}\left[\int_{-\infty}^{\infty} x(t)e^{-j\mu t}dt\right] dj\mu = \int_{-\infty}^{\infty} x(t)\left[\int_{j\omega}^{\infty} e^{-j\mu t}dj\mu\right] dt$$

$$= \int_{-\infty}^{\infty} x(t)\left[\frac{e^{-j\mu t}}{-t}\right]_{j\omega}^{\infty} dt = \int_{-\infty}^{\infty}\frac{x(t)}{t}e^{-j\omega t}dt = \mathcal{F}\left\{\frac{x(t)}{t}\right\}$$

□

$-END-$