# Properties of Fourier Transform II

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## Summary

| Name | Original | Transformed |
|---|---|---|
| General Time Integral | $\int_{-\infty}^{t} f(\tau)d\tau$ | $\frac{1}{j\omega}F(j\omega) + \pi\delta(j\omega)F(0)$ - |
| Convolution in Time | $f_1(t) * f_2(t)$ | $F_1(j\omega)\,F_2(j\omega)$ - |
| Convolution in Frequency | $f_1(t)f_2(t)$ | $\frac{1}{2\pi}F_1(j\omega) * F_2(j\omega)$ - |
| Bessel's Inequality | $\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx \geq \frac{a_0^2}{2} + \sum_{n=1}^{\infty}[a_n^2 + b_n^2]$ - | |
| Parseval Theorem | $\int_{-\infty}^{\infty}[f(x)]^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}[F(j\omega)]^2\,dx$ - | |

## 1 General Time Integration

$$\int_{-\infty}^{t} f(\tau)d\tau = \pi\delta(j\omega)F(0) + \frac{1}{j\omega}F(j\omega)$$

*Proof.* □

## 2 Convolution

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

### 2.1 Convolution is commutative : $f * g = g * f$

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

Let

$$t - \tau = t' \begin{cases} d\tau = -dt' \\ \tau = \infty \ \Rightarrow t' = -\infty \\ \tau = -\infty \ \Rightarrow t' = +\infty \end{cases}$$

$$f(t) * g(t) = \int_{\infty}^{-\infty} f(t-t')g(t')(-1)dt' = \int_{-\infty}^{\infty} f(t-t')g(t')dt' = g(t') * f(t') = g(t) * f(t)$$

## 2.2 Convolution is associative : $(f * g) * h = f * (g * h) = (f * h) * g$

# 3 $\mathcal{F}$

## 3.1 Convolution in time

$$f_1(t) * f_2(t) \longleftrightarrow F_1(j\omega)F_2(j\omega)$$

*Proof.*

$$\mathcal{F}\left\{f_1(t) * f_2(t)\right\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_1(\tau)f_2\left(t-\tau\right)d\tau\right]e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} f_1(\tau)\left[\int_{-\infty}^{\infty} f_2\left(t-\tau\right)e^{-j\omega t}dt\right]d\tau = \int_{-\infty}^{\infty} f_1(\tau)\left[F_2\left(j\omega\right)e^{-j\omega\tau}\right]d\tau$$

$$= F_2\left(j\omega\right)\int_{-\infty}^{\infty} f_1(\tau)e^{-j\omega\tau}d\tau = F_1\left(j\omega\right)F_2\left(j\omega\right)$$

□

## 3.2 Convolution in frequency

$$f_1\left(t\right)f_2\left(t\right) \longleftrightarrow \frac{1}{2\pi}F_1\left(j\omega\right) * F_2\left(j\omega\right)$$

*Proof.* Consider the RHS

$$\mathcal{F}\left\{f_1\left(t\right)f_2\left(t\right)\right\} = \int_{-\infty}^{\infty} f_1\left(t\right)f_2\left(t\right)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1\left(j\mu\right)e^{j\mu t}dj\mu\right]f_2(t)e^{-j\omega t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1\left(j\mu\right)\left[\underbrace{\int_{-\infty}^{\infty} f_2(t)e^{-j(\omega-\mu)t}dt}_{F_2(j(\omega-\mu))}\right]dj\mu$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1\left(j\mu\right)F_2\left(j\left(\omega-\mu\right)\right)dj\mu = \frac{1}{2\pi}F_1\left(j\omega\right) * F_2\left(j\omega\right)$$

□

## 3.3 Parseval's Theorem

$$\int_{-\infty}^{\infty}|f(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(j\omega)|^2dj\omega$$

### 3.3.1 Bessel's Inequality

Consider

$$\int_{-\pi}^{\pi}\left[f(x) - s_k(x)\right]^2dx \geq 0$$

And where

$$s_k(x) = \frac{a_0}{2} + \sum_{n=1}^{k} [a_n \cos nx + b_n \sin nx]$$

Expanding

$$\int_{-\pi}^{\pi} [f(x) - s_k(x)]^2 \, dx = \int_{-\pi}^{\pi} [f(x)]^2 \, dx - 2\int_{-\pi}^{\pi} f(x)s_k(x)dx + \int_{-\pi}^{\pi} [s_k(x)]^2 \, dx \geq 0$$

Consider the last term

$$\int_{-\pi}^{\pi} [s_k(x)]^2 \, dx = \int_{-\infty}^{\infty} \left[\frac{a_0}{2} + \sum_{n=1}^{k} [a_n \cos nx + b_n \sin nx]\right]^2 dx$$

$$= \int_{-\pi}^{\pi} \left[\frac{a_0^2}{4} + a_0 \sum_{n=1}^{k} [a_n \cos nx + b_n \sin nx] + \sum_{n=1}^{k} [a_n \cos nx + b_n \sin nx]^2\right] dx$$

$$= \frac{a_0^2}{4} \underbrace{\int_{-\pi}^{\pi} dx}_{2\pi} + \underbrace{a_0 \int_{-\pi}^{\pi} \sum_{n=1}^{k} [a_n \cos nx + b_n \sin nx] \, dx}_{0} + \int_{-\pi}^{\pi} \sum_{n=1}^{k} [a_n \cos nx + b_n \sin nx]^2 \, dx$$

$$= \frac{a_0^2 \pi}{2} + \int_{-\pi}^{\pi} \sum_{n=1}^{k} \left[a_n^2 \cos^2 nx + b_n^2 \sin^2 nx + \underbrace{2a_n b_n \sin nx \cos nx}_{0}\right] dx$$

$$= \frac{a_0^2 \pi}{2} + \sum_{n=1}^{k} \left[a_n^2 \pi + b_n^2 \pi\right] = \pi \left(\frac{a_0^2}{2} + \sum_{n=1}^{k} \left[a_n^2 + b_n^2\right]\right)$$

*Remark.* Orthogonal : $\int_{-\pi}^{\pi} \cos nx dx = \langle \cos nx, 1 \rangle = \int_{-\pi}^{\pi} \sin nx dx = \langle \sin nx, 1 \rangle \equiv 0 \ \forall n$

Then consider the middle term

$$2\int_{-\pi}^{\pi} f(x)s_k(x)dx = 2\int_{-\pi}^{\pi} f(x) \left[\frac{a_0}{2} + \sum_{n=1}^{k} [a_n \cos nx + b_n \sin nx]\right] dx$$

$$= a_0 \int_{-\pi}^{\pi} f(x)dx + 2\sum_{n=1}^{k} \left[\int_{-\pi}^{\pi} f(x)a_n \cos nx dx + \int_{-\pi}^{\pi} f(x)b_n \sin nx dx\right]$$

$$= a_0 (a_0 \pi) + 2\sum_{n=1}^{k} \left[a_n \underbrace{\int_{-\pi}^{\pi} f(x) \cos nx dx}_{a_n \pi} + b_n \underbrace{\int_{-\pi}^{\pi} f(x) \sin nx dx}_{b_n \pi}\right]$$

$$= \left\{a_0^2 \pi + 2\pi \sum_{n=1}^{k} \left[a_n^2 + b_n^2\right]\right\} = \pi \left(a_0^2 \pi + 2\sum_{n=1}^{k} \left[a_n^2 + b_n^2\right]\right)$$

Therefore,

$$\int_{-\pi}^{\pi} [f(x) - s_k(x)]^2 \, dx = \int_{-\pi}^{\pi} [f(x)]^2 \, dx - 2\int_{-\pi}^{\pi} f(x)s_k(x)dx + \int_{-\pi}^{\pi} [s_k(x)]^2 \, dx \geq 0$$

$$\iff \int_{-\pi}^{\pi} [f(x)]^2 \, dx - \pi \left(a_0^2 \pi + 2\sum_{n=1}^{k} \left[a_n^2 + b_n^2\right]\right) + \pi \left(\frac{a_0^2}{2} + \sum_{n=1}^{k} \left[a_n^2 + b_n^2\right]\right) \geq 0$$

$$\iff \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx - \sum_{n=1}^{k}\left[a_n^2+b_n^2\right] - \frac{a_0^2}{2} \geq 0$$

$$\iff \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx \geq \frac{a_0^2}{2} + \sum_{n=1}^{k}\left[a_n^2+b_n^2\right]$$

This inequality is true for all $k$ (as LHS is independent of $k$)

Thus the **Bessel's Inequality**

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx \geq \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left[a_n^2+b_n^2\right]$$

And, since the $\int_{-\pi}^{\pi}[f(x)-s_k(x)]^2\,dx$ is the error term, divide it with $2\pi$ to form the *mean square error*

$$MSE = \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)-s_k(x)]^2\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx - \frac{1}{2\pi}\cdot 2\int_{-\pi}^{\pi}f(x)s_k(x)dx + \frac{1}{2\pi}\int_{-\pi}^{\pi}[s_k(x)]^2\,dx \geq 0$$

$$\iff \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx \geq \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{k}\left[a_n^2+b_n^2\right]$$

### 3.3.2 Parseval Equality

As LHS $\geq$ RHS, that means, LHS is the upper bound of the RHS

And, the RHS is monotonic increaseing

Thus

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left[a_n^2+b_n^2\right] \qquad \text{Converge}$$

For convergence,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$$

Further more , Fourier series converge when

$$\lim_{k\to\infty}\int_{-\pi}^{\pi}\left\{f(x) - \left[\frac{a_0}{2} + \sum_{n=1}^{k}\left[a_n\cos nx + b_n\sin nx\right]\right]\right\}^2 dx = 0$$

If fourier series converge , then it is **Parseval's Eequality**

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left[a_n^2+b_n^2\right]$$

Recall, $a^2+b^2=c^2$ , which is the magnitude of the frequency spectra

*Remark.* General Bessel's Equality

$$\sum_{k=1}^{\infty}\left|\left\langle x, e_k\right\rangle\right|^2 = \left\|x\right\|^2$$

$\left\langle\ \right\rangle$ is dot product

### 3.4 Another Form of Parseval's Eequality

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} f(x) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} dj\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) \left[ \int_{-\infty}^{\infty} f(x) e^{j\omega t} dt \right] dj\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) \left[ \int_{-\infty}^{\infty} f(x) e^{-j(-\omega)t} dt \right] dj\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) F\left(-j\omega\right) dj\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) F * \left(j\omega\right) dj\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 dj\omega$$

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