Laplace Transform Pairs - I

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1 Summary

	$f(t)$	$F(s)$
Unity	1	$\frac{1}{s}$
Polynomial	t^1 $t^2 \quad n \in \mathbb{Z}^+$ t^n	${}^1\!/_{s^2}$ ${}^1\!/_{s^3}$ $\frac{n!}{s^{n+1}}$
Exponential	e^{at}	$\frac{1}{s-a}$
Complex Exponential	$e^{j\omega t}$	$\frac{1}{s-j\omega}$
Sine	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
Cosine	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
Damped sine	$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
Damped cosine	$e^{at} \cos \omega t$	$\frac{s}{(s-a)^2+\omega^2}$
Hyperbolic sine	$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
Hyperbolic cosine	$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$

2 The LT Pairs

2.1 Polynomials

2.1.1 Unity $\mathcal{L}\{1\}$

$$\mathcal{L}\{1\} = \int_0^\infty 1 \cdot e^{-st}dt = \frac{-1}{s}\int_0^\infty de^{-st} = \frac{-1}{s}\left[e^{-st}\right]_0^\infty = \frac{1}{s}$$

2.1.2 Polynomial $\mathcal{L}\{t^n\}$

$$\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st}dt = \frac{-1}{s}\int_0^\infty t^n de^{-st} = \frac{-1}{s}\left[t^n e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt^n$$

Recall, the limit $\lim_{N\to\infty}\frac{t^N}{e^N} = \lim_{N\to\infty}\frac{Nt^{N-1}}{e^N} = ... = 0$ (**L'Hospital Rule**)

$$= \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt$$

Then,

$$= \frac{n}{s}\cdot\frac{n-1}{s}\int_0^\infty t^{n-2}e^{-st}dt = ... = \frac{n!}{s^n}\mathcal{L}\{1\} = \frac{n!}{s^{n+1}}$$

2.2 Exponential $\mathcal{L}\{e^\theta\}$

2.2.1 Real Exponential

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-(s-a)t}dt = \frac{-1}{s-a}\underbrace{\left[e^{-(s-a)t}\right]_0^\infty}_{-1} = \frac{1}{s-a}$$

2.2.2 Complex Exponential

$$\mathcal{L}\{e^{j\omega t}\} = \mathcal{L}\{e^{at}\}_{a=j\omega} = \left(\frac{1}{s-a}\right)_{a=j\omega} = \frac{1}{s-j\omega}$$

2.2.3 SinCos

Find $\mathcal{L}\{\sin\omega t\}$ by integration-by-parts

$$\mathcal{L}\{\sin\omega t\} = \int_0^\infty \sin(\omega t)e^{-st}dt = \frac{-1}{s}\int_0^\infty \sin(\omega t)de^{-st} = \underbrace{\left[\frac{-1}{s}\sin(\omega t)e^{-st}\right]_0^\infty}_{0} + \frac{\omega}{s}\int_0^\infty \cos(\omega t)e^{-st}dt$$

$$= \frac{\omega}{-s^2}\int_0^\infty \cos(\omega t)de^{-st} = \frac{\omega}{-s^2}\underbrace{\left(\cos\omega t e^{-st}\right)_0^\infty}_{-1} - \frac{\omega^2}{s^2}\int_0^\infty \sin\omega t e^{-st}dt = \frac{\omega}{s^2} - \frac{\omega^2}{s^2}\mathcal{L}\{\sin\omega t\}$$

$$\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2} - \frac{\omega^2}{s^2}\mathcal{L}\{\sin\omega t\} \qquad \Rightarrow \quad \mathcal{L}\{\sin\omega t\} = \frac{\omega/s^2}{1+\omega^2/s^2} = \frac{\omega}{s^2+\omega^2}$$

Find $\mathcal{L}\{\cos\omega t\}$ by integration-by-parts

$$\mathcal{L}\{\cos\omega t\} = \int_0^\infty \cos(\omega t)e^{-st}dt = \frac{-1}{s}\int_0^\infty \cos(\omega t)de^{-st} = \underbrace{\left[\frac{-1}{s}\cos(\omega t)e^{-st}\right]_0^\infty}_{\frac{1}{s}} - \frac{\omega}{s}\int_0^\infty \sin(\omega t)e^{-st}dt$$

$$= \frac{1}{s} + \frac{\omega}{s^2}\int_0^\infty \sin(\omega t)de^{-st} = \frac{1}{s} + \frac{\omega}{s^2}\underbrace{\left(\sin\omega t \cdot e^{-st}\right)_0^\infty}_{0} - \frac{\omega^2}{s^2}\int_0^\infty \cos\omega t e^{-st}dt = \frac{1}{s} - \frac{\omega^2}{s^2}\mathcal{L}\left\{\cos\omega t\right\}$$

$$\mathcal{L}\{\cos\omega t\} = \frac{1}{s} - \frac{\omega^2}{s^2}\mathcal{L}\left\{\cos\omega t\right\} \qquad \Rightarrow \quad \mathcal{L}\left\{\cos\omega t\right\} = \frac{1/s}{1 + \omega^2/s^2} = \frac{s}{s^2+\omega^2}$$

Using such direct integration method is very clumsy and error prone!!

Method 2 : $\mathcal{L}\{e^{j\omega t}\} = \mathcal{L}\{\mathbf{cos}\omega t + j\mathbf{sin}\omega t\}$

$$\mathcal{L}\{e^{j\omega t}\} = \frac{1}{s-j\omega} = \frac{1}{s-j\omega}\cdot\frac{s+j\omega}{s+j\omega} = \frac{s}{s^2+\omega^2} + j\frac{\omega}{s^2+\omega^2}$$

$$\therefore \mathcal{L}\{\cos\omega t\} = Re\left[\mathcal{L}\{e^{j\omega t}\}\right] = \frac{s}{s^2+\omega^2} \qquad \mathcal{L}\{\sin\omega t\} = Im\left[\mathcal{L}\{e^{j\omega t}\}\right] = \frac{\omega}{s^2+\omega^2}$$

2.2.4 Damped Sin-Cos

Damped : $Re(a) < 0$

Consider $\mathcal{L}\{e^{at}e^{j\omega t}\}$

$$\mathcal{L}\{e^{at}e^{j\omega t}\} = \int_0^\infty e^{(j\omega+a)t}e^{-st}dt = \int_0^\infty e^{-(s-j\omega-a)t}dt = \frac{1}{s-a-j\omega}$$

$$= \frac{1}{s-a-j\omega}\cdot\frac{s-a+j\omega}{s-a+j\omega} = \frac{(s-a)+j\omega}{(s-a)^2+\omega^2}$$

So

$$\mathcal{L}\{e^{at}\sin\omega t\} = Re\mathcal{L}\{e^{at}e^{j\omega t}\} = \frac{s-a}{(s-a)^2+\omega^2} \quad \mathcal{L}\{e^{at}\sin\omega t\} = Im\mathcal{L}\{e^{at}\sin\omega t\} = \frac{\omega}{(s-a)^2+\omega^2}$$

2.2.5 Hyperbolic SinCos $\mathcal{L}\{\mathbf{sinh}\omega t\}$ $\mathcal{L}\{\mathbf{cosh}\omega t\}$

$$\mathcal{L}\{\sinh\omega t\} = \mathcal{L}\{\sin(j\omega t)\} = \int_0^\infty \frac{e^{\omega t} - e^{-\omega t}}{2}\cdot e^{-st}dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-\omega)t} - e^{-(s+\omega)t}dt = \frac{1}{2}\left(\frac{1}{s-\omega} - \frac{1}{s+\omega}\right) = \frac{\omega}{s^2-\omega^2}$$

$$\mathcal{L}\{\cosh\omega t\} = \mathcal{L}\{\cos(j\omega t)\} = \int_0^\infty \frac{e^{\omega t} + e^{-\omega t}}{2}\cdot e^{-st}dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-\omega)t} + e^{-(s+\omega)t}dt = \frac{1}{2}\left(\frac{1}{s-\omega} + \frac{1}{s+\omega}\right) = \frac{s}{s^2-\omega^2}$$

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