$\mathcal{Z}$-transform properties

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$$X(z) = \mathcal{Z}\left\{x[n]\right\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Summary

	Original	Transformed
Linearity	$ax[n] + by[n]$	$aX(z) + bY(z)$
Delay	$x[n-1]$ $x[n-k]$	$z^{-1}X(z)$ $z^{-n}X(z)$
Hasten	$x[n+1]$ $x[n+2]$ $x[n+k]$	$zX(z) - zx(0)$ $z^2X(z) - z^2x(0) - zx(1)$ $z^kX(z) - z^kx(0) - z^{k-1}x(1) - ... - zx(k-1)$
Multiplication	$n \cdot x[n]$ $n^k \cdot x[n]$	$-z\frac{d}{dz}X(z)$ $(-1)^k\frac{d^k}{dz^k}X(z)$
Convolution	$x[n] * y[n]$	$X(z) \cdot Y(z)$
Convolution	$x[n] \cdot y[n]$	$X(z) * Y(z)$
Initial Value	$x[0] = \lim_{z \mapsto \infty} X(z)$	

1 The $\mathcal{Z}$-Transform

1.1 Linear $ax[n] + by[n] \longleftrightarrow aX[z] + bY[z]$

Proof. Direct

$$\mathcal{Z}\left\{ax[n] + by[n]\right\} = \sum_{n=-\infty}^{\infty}\left\{ax[n] + by[n]\right\}z^{-n} = a\sum_{n=-\infty}^{\infty} x[n]z^{-n} + b\sum_{n=-\infty}^{\infty} y[n]z^{-n} = aX(z) + bY(z)$$

□

1.2 Delay $x[n-k] \longleftrightarrow z^{-k}X(z)$

Proof. Use the substitution $n - 1 = m$ $n = m + 1, n = \pm\infty \rightarrow m = \pm\infty$

$$\mathcal{Z}\left\{x[n-1]\right\} = \sum_{n=-\infty}^{\infty} x[n-1]z^{-n} = \sum_{m=-\infty}^{\infty} x[m]z^{-(m+1)} = z^{-1}\sum_{m=-\infty}^{\infty} x[m]z^{-m} = z^{-1}X(z)$$

Same for $n - k$ □

1.3 Hasten $x[n+k]$

Consider $x[n+1]$, let $n+1=m$, $n=m-1$, $n=\pm\infty \Rightarrow m=\pm\infty$

$$\mathcal{Z}\left\{x[n+1]\right\} = \sum_{n=-\infty}^{\infty} x[n+1]z^{-n} = \sum_{m=-\infty}^{\infty} x[m]z^{-(m-1)} = z\sum_{m=-\infty}^{\infty} x[m]z^{-m} = zX(z)$$

1.4 Mutiplication $n \cdot x[n] \longleftrightarrow -z\dfrac{dX(z)}{dz}$

Consider $\dfrac{d}{dz}X(z)$

$$\frac{d}{dz}X(z) = \frac{d}{dz}\mathcal{Z}\left\{x[n]\right\} = \frac{d}{dz}\sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n]\frac{d}{dz}z^{-n} = -\sum_{n=-\infty}^{\infty} nx[n]z^{-n-1}$$

$$-z\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = \mathcal{Z}\left\{n \cdot x[n]\right\}$$

Repeat the process,

$$n^k \cdot x[n] \longleftrightarrow (-1)^k z^k \frac{d^k}{dz^k}X(z)$$

1.5 Convolution $x[n] * y[n] \longleftrightarrow X(z)Y(z)$ and $x[n]y[n] \longleftrightarrow X(z) * Y(z)$

$$\mathcal{Z}\left\{x[n] * y[n]\right\} = \sum_{n=-\infty}^{\infty} \left\{x[n] * y[n]\right\} z^{-n} = \sum_{n=-\infty}^{\infty} \left\{\sum_{k=0}^{M} x[k]y[n-k]\right\} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left\{\sum_{k=0}^{M} x[k]y[n-k]\right\} z^{-n}$$

1.6 Inital Value $x[0] = \lim_{z\mapsto\infty} X(z)$

Proof.

$$\mathcal{Z}\left\{x[n]\right\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z)$$

For *causal* signal, $x[n] = 0 \ \forall n < 0$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + ...$$

$$X(z) = x[0] + z^{-1}\left(x[1] + x[2]z^{-1} + ...\right)$$

Take limit

$$\lim_{z\mapsto\infty} X(z) = \lim_{z\mapsto\infty} x[0] + \lim_{z\mapsto\infty} z^{-1}\left(x[1] + x[2]z^{-1} + ...\right)$$

$\therefore$

$$x[0] = \lim_{z \mapsto \infty} X(z)$$

$\square$

Remark. 1. If the limit exists, 2. Causal Signal

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