Laplace Transform Pairs II

Ang Man Shun

November 13, 2012

1 Summary

	$f(t)$	$F(s)$
Dirac Delta Impulse	$\delta(t)$	1
Delay Impulse	$\delta(t-\tau)$	$e^{-s\tau}$
Heaviside Unit Step	$u(t)$	$\frac{1}{s}$
Sign Function	$\text{sgn}(t)$	$\frac{2}{s}$
Ramp	$r(t)$	$\frac{1}{s^2}$
General Power	$t^a \quad a \in \mathbb{R}$	$\frac{\Gamma(a+1)}{s^{a+1}}$
Natural Log	$\ln(t)$	$-\frac{\gamma + \ln s}{s}$ or $\frac{\Gamma'(1) - \ln s}{s}$
Two-side exponential decay	$e^{-a\|t\|}$	$\frac{2a}{a^2 - s^2}$

2 The LT Pairs

2.1 Dirac Delta Impulse and Delay Impulse $\mathcal{L}\{\delta(t)\}$ $\mathcal{L}\left\{\delta(t-\tau)\right\}$

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases} \qquad \int_{\mathbb{R}} \delta(t)dt = \int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\delta(t-\tau) = \begin{cases} \infty & t = \tau \\ 0 & t \neq \tau \end{cases}$$

$$\mathcal{L}\{\delta(t)\} = \int_0^{\infty} \delta(t)e^{-st}dt = \int_0^{\infty} \delta(t)e^{0}dt = \int_0^{\infty} \delta(t)dt = 1$$

$$\mathcal{L}\{\delta(t-\tau)\} = \int_0^\infty \delta(t-\tau)e^{-st}dt$$

Let $t-\tau = T$, $dt = dT$, $t=\infty, T=\infty$, $t=0$, $T=-\tau$, but $\delta(t)=0$ for $t=-\tau$, so

$$= \int_0^\infty \delta(T)e^{-s(T+\tau)}dT = e^{-s\tau}\int_0^\infty \delta(T)dT = e^{-s\tau}$$

- $\delta(t)$ is a impulse with ∞ height with 0 width at $t=0$
- $\delta(t-\tau)$ is translation version of $\delta(t)$ at $t=\tau$

2.2 Heavisde Unit Step Function $\mathcal{L}\{u(t)\}$

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad u(t) = \int_{-\infty}^{\infty} \delta(t)dt$$

$$\mathcal{L}\{u(t)\} = \int_0^\infty u(t)e^{-st}dt = \int_0^\infty e^{-st}dt = \frac{1}{s}$$

- Effect of unit step is to change range of a function from $\Re_0$ *on to* $[0,\infty)$
- Part of $f(t)$ that $t<0$ is ignored :$\mathcal{L}\{u(t)f_1(t)\} = \mathcal{L}\{f_2(t)\}$, $f_1(t) : D_1 \longmapsto R_1$ $f_2 : D_1 \longmapsto [0\,,\,\infty]$
- General One Sided Laplace Transfrom for any function : $\mathcal{L}\{f(t)\} = \int_0^\infty f(t)u(t)e^{-st}dt$
- General One Sided Laplace Transform for any function start at

  t_0 is $\mathcal{L}\{u(t_0-t)f(t)\} = \int_0^\infty f(t)u(t_0-t)e^{-st}dt$

2.3 Sign Function $\mathcal{L}\{\text{sgn}(t)\}$

$$\text{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

$$\mathcal{L}\{\text{sgn}t\} = \int_{-\infty}^{+\infty} \text{sgn}t\, e^{-st}dt = \int_0^\infty e^{-st}dt - \int_{-\infty}^0 e^{-st}dt = 2\int_0^\infty e^{-st}dt = \frac{2}{s}$$

2.4 Ramp Function $\mathcal{L}\{r(t)\}$

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad r(t) = \int_{-\infty}^{\infty} u(t)dt$$

$$\mathcal{L}\{r(t)\} = \int_0^\infty r(t)e^{-st}dt = \int_0^\infty te^{-st}dt = \frac{1}{s^2}$$

2.5 Power $\mathcal{L}\{t^a\}$

$$\mathcal{L}\{t^a\} = \int_0^\infty t^a e^{-st} dt$$

Let $u = st \longrightarrow \begin{cases} t = \dfrac{u}{s} \\ dt = \dfrac{du}{s} \end{cases}$ $$= \int_0^\infty \left(\frac{u}{s}\right)^a e^{-u} \frac{du}{s} = \frac{1}{s^{a+1}} \int_0^\infty u^a e^{-u} du = \frac{\Gamma(a+1)}{s^{a+1}}$$

Gamma Function $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$

2.6 Natural Log $\mathcal{L}\{\ln t\}$

$$\mathcal{L}\{\ln t\} = \int_0^\infty \ln t \cdot e^{-st} dt = ???$$

Direct method does not apply, consider $\mathcal{L}\{t^a\} = \int_0^\infty t^a e^{-st} dt$, and recall that

$$\frac{dt^a}{da} = \frac{de^{\ln t^a}}{da} = \frac{de^{a \ln t}}{da} = \ln t \cdot t^a$$

So

$$\frac{d}{da}\mathcal{L}\{t^a\} = \frac{d}{da}\int_0^\infty t^a e^{-st} dt = \frac{d}{da}\left(\frac{\Gamma(a+1)}{s^{a+1}}\right)$$

$$\iff \int_0^\infty \ln t \cdot t^a e^{-st} dt = \frac{\left(\frac{d}{da}\Gamma(a+1)\right) s^{a+1} - \left(\frac{d}{da}s^{a+1}\right)\Gamma(a+1)}{\left(s^{a+1}\right)^2}$$

$$\iff \int_0^\infty \ln t \cdot t^a e^{-st} dt = \frac{\Gamma'(a+1)s^{a+1} - \ln s \cdot s^{a+1}\Gamma(a+1)}{\left(s^{a+1}\right)^2} = \frac{\Gamma'(a+1) - \ln s \cdot \Gamma(a+1)}{s^{a+1}}$$

Put $a = 0$

$$\int_0^\infty \ln t \cdot e^{-st} dt = \frac{\Gamma'(1) - \ln s \cdot \Gamma(1)}{s^1} = \frac{\Gamma'(1) - \ln s}{s} = -\frac{\gamma + \ln s}{s}$$

$\Gamma'(1) = -\gamma$, the $Euler - Mascheroni\ constant$

$\gamma = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \log n = 05772...$ It is not yet known it is rational or irrational!!

2.7 2-side exponential decay $\mathcal{L}\left\{e^{-a|t|}\right\}$

$$\mathcal{L}\left\{e^{-a|t|}\right\} = \int_{-\infty}^\infty e^{-a|t|}e^{-st} dt = \int_{-\infty}^0 e^{at}e^{-st} dt + \int_0^\infty e^{-at}e^{-st} dt = \int_{-\infty}^0 e^{-(s-a)t} dt + \int_0^\infty e^{-(s+a)t} dt$$

$$= -\int_{-\infty}^0 e^{-(a-s)(-t)}(-dt) + \frac{\left[e^{-(s+a)t}\right]_0^\infty}{-(s+a)} = -\int_{+\infty}^0 e^{-(a-s)\tau} d\tau + \frac{1}{(s+a)} = \int_0^\infty e^{-(a-s)\tau} d\tau + \frac{1}{(s+a)}$$

$$= \frac{\left[e^{-(a-s)t}\right]_0^\infty}{-(a-s)} + \frac{1}{(s+a)} = \frac{1}{a-s} + \frac{1}{a+s} = \frac{2a}{a^2 - s^2}$$

–END–