Hilbert Transform

Ang Man Shun

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Reference Hwei Hsu , *Analog and Digital Communications*

1 The Hilbert Transform

$$\widehat{x}(t) = \mathcal{H}\left\{x(t)\right\} = x(t) * \frac{1}{\pi t} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau$$

- $\widehat{x}(t) \in \mathbb{R}$
- Hilber Transform can be treated as convolution of $x(t)$ with $\dfrac{1}{\pi t}$, which is $h(t)$
- Hilber Transform can be treat as a $\pm\dfrac{\pi}{2}$ orpeator : $\mathcal{H}\left\{\cos\omega t\right\} = \sin\omega t = \cos\left(\dfrac{\pi}{2} - \omega t\right)$
- A real function $x(t)$ and its Hilbert Transfrom can form a *analytic signal* : $x_{+}(t) = x(t) + j\widehat{x}(t)$
- $\dfrac{x(\tau)}{t-\tau}$ has a pole when $\tau = t$, but when using the Principal value, the integral exists

2 Properties

2.1 Linear Operator

$$\mathcal{H}\left\{ax(t) + by(t)\right\} = \mathcal{H}\left\{ax(t)\right\} + \mathcal{H}\left\{by(t)\right\} = a\mathcal{H}\left\{x(t)\right\} + b\mathcal{H}\left\{y(t)\right\} = a\widehat{x}(t) + b\widehat{y}(t)$$

Proof. By direct proof □

2.2 Fourier Transform of Hilbert Transform

$$x(t) \overset{\mathcal{H}}{\longleftrightarrow} \widehat{x}(t) \qquad x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

$$\mathcal{F}\left\{\mathcal{H}\left\{x(t)\right\}\right\} = \mathcal{F}\left\{\widehat{x}(t)\right\} = \mathcal{F}\left\{x(t) * h(t)\right\} = X(\omega)H(\omega)$$

$$H(\omega) = \mathcal{F}\left\{h(t)\right\} = \mathcal{F}\left\{\frac{1}{\pi t}\right\}$$

By Fourier Paris of sign function and duality property,

$$\operatorname{sgn}(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{2}{j\omega} \qquad \Rightarrow \qquad \operatorname{sgn}(\omega) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} \cdot \frac{2}{jt}$$

$\therefore$

$$h(t) = \frac{1}{\pi t} \xleftrightarrow{\mathcal{F}} -j\text{sgn}(\omega) = H(\omega)$$

Therefore, the Fourier Transform of the Hilbert Transform is

$$\widehat{X}(\omega) = -j\text{sgn}\,(\omega)\, X(\omega)$$

i.e.

$$(\mathcal{F} \cdot \mathcal{H})\, x(t) = -j\text{sgn}(\omega)\mathcal{F}\,\{x(t)\}$$

2.3 Skew-Involution / Anti-involution

With the help of Fourier Transform, some interesting property of Hilbert Transform can be found

$$x(t) \xleftrightarrow{\mathcal{H}} \widehat{x}(t) \qquad x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \quad \widehat{x}(t) \xleftrightarrow{\mathcal{F}} \widehat{X}(\omega) = -j\text{sgn}(x)X(\omega)$$

Consider

$$\widehat{\widehat{x}}(t) \xleftrightarrow{\mathcal{F}} \widehat{\widehat{X}}(\omega) = (-j\text{sgn}(\omega))\left(\widehat{X}(\omega)\right) = (-j\text{sgn}(\omega))\,(-j\text{sgn}(x)X(\omega)) = (-j\text{sgn}(\omega))^2\, X(\omega)$$

$$(-j\text{sgn}(\omega))^2 = (-j)^2\text{sgn}^2(\omega) = -1$$

So

$$\widehat{\widehat{x}}(t) \xleftrightarrow{\mathcal{F}} \widehat{\widehat{X}}(\omega) = -X(\omega)$$

i.e.

$$\widehat{\widehat{X}}(\omega) = -X(\omega) \qquad \Longleftrightarrow \mathcal{H}^2 X(\omega) = -X(\omega)$$

$$\Longleftrightarrow \mathcal{H}^2 = -I$$

And thus

$$\mathcal{H}^{-1} = -\mathcal{H} \qquad\qquad \mathcal{H}^4 = I$$

2.4 Orthogonal

$$\langle x(t)\,,\, \mathcal{H}x(t)\rangle = 0$$

Proof. By Parseval's Theorem in Fourier Analysis □

$$\int_{-\infty}^{\infty} x(t)\widehat{x}(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\widehat{X}(-\omega)d\omega$$

For $x(t) \in \mathbb{R}$,

$$\widehat{X}(-\omega) = -j\text{sgn}\,(-\omega)\, X(-\omega) = j\text{sgn}(\omega)X*(\omega)$$

Plug into the dot prodcut

$$\int_{-\infty}^{\infty} x(t)\widehat{x}(t)dt = \frac{j}{2\pi}\int_{-\infty}^{\infty} \text{sgn}(\omega)X(\omega)X*(\omega)d\omega = \frac{j}{2\pi}\int_{-\infty}^{\infty} \text{sgn}(\omega)\,|X(\omega)|^2\, d\omega$$

$\text{sgn}(\omega)\,|X(\omega)|^2$ is odd function , as $|X(\omega)|^2$ is even, $\text{sgn}(\omega)$ is odd, and thus the integral is zero.

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