Properties of Laplace Transform - I

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2012-8-14

Reference C.K. Alexander , M.N.O Sadiku *Fundamentals of Electric Circuits*

Summary

	t-domain function	s-domain function
1. Linear	$af_1(t) + bf_2(r)$	$aF_1(s) + bF_1(s)$
2. Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
3. Time Shift	$f(t-t_0)\,u(t-t_0)$	$e^{-st_0}F(s)$
4. Frequency Shift	$e^{at}f(t)$	$F(s-a)$
5. Reverse Time	$f(-t)$	$F(-s)$
6. Time Differentiation	$\frac{df(t)}{dt}$ $\frac{d^nf(t)}{dt^n}$	$sF(s) - f(0)$ $s^nF(s) - s^{n-1}f(0) - ... - f^{(n-1)}(0)$
7. Time Integral	$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$
8. Frequency Differentiation t-multiplication	$tf(t)$ $t^nf(t)$	$-\frac{dF(s)}{ds}$ $(-1)^n\frac{d^nF(s)}{ds^n}$
9. t-division Frequency Integration	$\frac{f(t)}{t}$	$\int_s^\infty F(u)du$
10. Periodic	$f(t)$	$\frac{F_0(s)}{1-e^{-Ts}}$
11. Initial Value	$f(0)$	$\lim_{s\to\infty} sF(s)$
12. Final Value	$f(\infty)$	$\lim_{s\to 0} sF(s)$

1 $\mathcal{L}$

1.1 Linearity

$$ax + by \leftrightarrow aX + bY$$

Proof.

$$\mathcal{L}\{ax(t) + by(t)\} = \int_0^\infty (ax(t) + by(t))\,e^{-st}dt = a\int_0^\infty x(t)e^{-st} + b\int_0^\infty y(t)e^{-st}dt = aX(s) + bY(s)$$

□

1.2 Scaling

$$x(at) \longleftrightarrow \frac{1}{|a|}X\left(\frac{s}{a}\right) \quad x(at) \longleftrightarrow \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

Proof.

$$\mathcal{L}\{x(at)\} = \int_0^\infty x(at)e^{-st}dt = \int_0^\infty x(\tau)e^{-s\frac{\tau}{a}}\frac{d\tau}{|a|} = \int_0^\infty x(\tau)e^{-\frac{s}{a}\tau}\frac{d\tau}{|a|} = \frac{1}{|a|}X\left(\frac{s}{a}\right)$$

□

1.3 Modulation in Time / Time-Shift

$$\mathcal{L}\left\{x\left(t-t_0\right)u\left(t-t_0\right)\right\} = X(s)e^{-st_0}$$

Proof.

$$\mathcal{L}\left\{x\left(t-t_0\right)u\left(t-t_0\right)\right\} = \int_{t_0}^\infty x(t-t_0)e^{-st}dt = \int_0^\infty x(\tau)e^{-s(\tau+t_0)}d\tau = e^{-st_0}\int_0^\infty x(\tau)e^{-s\tau}d\tau = e^{-st_0}X(s)$$

□

Remark. Heaviside Unit Step Function is used to keep the causality

1.4 Modulation in Frequency / Frequency Shift

$$e^{at}x(t) \longleftrightarrow X(s-a)$$

Proof. □

$$\mathcal{L}\{x(t)e^{at}\} = \int_0^\infty x(t)e^{at}e^{-st}dt = \int_0^\infty x(t)e^{-(s-a)t}dt = X(s-a)$$

1.5 Time-Reverse

$$x(-t) \longleftrightarrow X(-s)$$

Proof.

$$\mathcal{L}\{x(-t)\} = \int_0^\infty x(-t)e^{-st}dt = -\int_\infty^0 x(\tau)e^{s\tau}d\tau = \int_0^\infty x(\tau)e^{-(-s)\tau}d\tau = X(-s)$$

□

1.6 Time Differentiation

$$\frac{d}{dt}x(t) \longleftrightarrow sX(s) - x(0)$$

Proof.

$$\mathcal{L}\{\frac{dx(t)}{dt}\} = \int_0^\infty \frac{dx(t)}{dt}e^{-st}dt = \int_0^\infty e^{-st}dx(t)$$

$$= \underbrace{\left[e^{-st}x(t)\right]_0^\infty}_{-x(0)} - \underbrace{\int_0^\infty x(t)de^{-st}}_{sX(s)} = sX(s) - x(0)$$

□

Remark. Repeat ,

$$\mathcal{L}\left\{\frac{d}{dt}\left(\frac{dx(t)}{dt}\right)\right\} = s\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} - \frac{dx(t)}{dt}|_{t=0}$$

$$= s\left\{sX(s) - x(0)\right\} - \dot{x}(0) = s^2X(s) - sx(0) - \dot{x}(0)$$

The general form, which can be prove by $Mathematical\ Induction$, is

$$\mathcal{L}\left\{\frac{d^nx(t)}{dt^n}\right\} = s^nX(s) - s^{n-1}x(0) - ... - x^{(n-1)}(0)$$

1.7 Time Integration

$$\int_0^t x(\tau)d\tau \longleftrightarrow \frac{X(s)}{s}$$

Proof.

$$\mathcal{L}\left\{\int_0^t x(\tau)d\tau\right\} = \int_0^\infty \left\{\int_0^t x(\tau)d\tau\right\} e^{-st}dt = \int_0^\infty \left\{\int_0^t x(\tau)d\tau\right\} \left(\frac{de^{-st}}{-s}\right)$$

By Parts

$$= \underbrace{\left(\frac{e^{-st}\int_0^t x(\tau)d\tau}{-s}\right)_0^\infty}_{0} + \frac{1}{s}\int_0^\infty e^{-st}d\left\{\int_0^t x(\tau)d\tau\right\}$$

By Fundamental Theorem of Calculus ,

$$\frac{d}{dt}\left\{\int_0^t x(\tau)d\tau\right\} = x(t) \Rightarrow d\left\{\int_0^t x(\tau)d\tau\right\} = x(t)dt$$

The Laplace Transform then becomes

$$= \frac{1}{s}\int_0^\infty e^{-st}x(t)dt = \frac{X(s)}{s}$$

□

1.8 Frequency Differentiation / t-multiplication

$$t^n f(t) \longleftrightarrow (-1)^n \frac{d^n}{ds^n} F(s)$$

Proof. Consider $\frac{dF}{ds}$:

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt = \int_0^\infty f(t)\frac{de^{-st}}{ds}dt = -\int_0^\infty [tf(t)]\, e^{-st}dt = -\mathcal{L}\{tf(t)\}$$

$$\frac{d^n F(s)}{ds^n} = \int_0^\infty f(t)\left(\frac{d^n}{ds^n}e^{-st}\right)dt = (-1)^n\int_0^\infty t^n f(t)e^{-st}dt$$

□

1.9 Frequency Integration / t-division

$$\mathcal{L}\left\{\frac{x(t)}{t}\right\} = \int_s^\infty X(\mu)d\mu$$

Proof. Consider the RHS

$$\int_s^\infty X(\mu)d\mu = \int_s^\infty\left[\int_0^\infty x(t)e^{-\mu t}dt\right]d\mu = \int_0^\infty x(t)\left[\int_s^\infty e^{-\mu t}d\mu\right]dt$$

$$= \int_0^\infty \frac{x(t)}{-t}\left[e^{-\mu t}\right]_s^\infty dt = \int_0^\infty \frac{x(t)}{t}e^{-st}dt = \mathcal{L}\left\{\frac{x(t)}{t}\right\}$$

□

1.10 Time Periodicity

$$F(s) = \frac{F_0(s)}{1-e^{-Ts}}$$

Proof. Consider a periodic function $f(t)$ with period T

$$f(t) = \sum f_k(t) = f_0(t) + f_1(t) + ...$$

Where $f_k(t)$ is the k^{th} repetition of $f_0(t)$

$$f_1(t) = f_0(t-T)\,u(t-T) = f_0(t-1\cdot T)\,u(t-1\cdot T)$$

$$f_2(t) = f_0(t-2T)\,u(t-2T) \qquad f_k(t) = f_0(t-kT)\,u(t-kT)$$

$$f_0(t) = f_0(t)\,[u(t)-u(t-T)] \iff f_0(t) = \begin{cases} f_0(t) & 0<t<T \\ 0 & \text{else}\end{cases}$$

Then the Laplace Transform is

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\sum_{k=0}^\infty f_k(t)\right\} = \sum_{k=0}^\infty \mathcal{L}\{f_k(t)\} = \sum_{k=0}^\infty \mathcal{L}\{f_0(t-kT)u(t-kT)\} = \sum_{k=0}^\infty\left(F_0(t)e^{-kT}\right)$$

$$= F_0(t)\sum_{k=0}^\infty\left(e^{-kT}\right) = \frac{F_0(t)}{1-e^{-Ts}}$$

□

Remark. Geometric Series

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + ... = \frac{1}{1-x} \qquad |x| < 1$$

1.11 Initial Value

$$x(0) = \lim_{s\to\infty} sX(s)$$

Proof. Consider $\frac{dx(t)}{dt} \longleftrightarrow sX(s) - x(0)$

$$\int_0^{\infty} \frac{dx(t)}{dt} e^{-st} dt = sX(s) - x(0)$$

Take $\lim\limits_{s\to\infty}$ on both side,

$$\underbrace{\lim_{s\to\infty} \int_0^{\infty} \frac{dx(t)}{dt} e^{-st} dt}_{0} = \lim_{s\to\infty} sX(s) - x(0)$$

$$\underbrace{x(0)}_{\text{t-domain}} = \underbrace{\lim_{s\to\infty} sX(s)}_{\text{s-domain}}$$

□

1.12 Final Value

$$x(\infty) = \lim_{s\to 0} sX(s)$$

Proof. Take $\lim\limits_{s\to 0}$ on both side,

$$\underbrace{\lim_{s\to 0} \int_0^{\infty} \frac{dx(t)}{dt} e^{-st} dt}_{\int_0^{\infty} \mathrm{d}x(t)t = x(\infty) - x(0)} = \lim_{s\to\infty} sX(s) - x(0)$$

$$x(\infty) = \lim_{s\to\infty} sX(s)$$

$$-END-$$

□