If $\mathbf{A}$ and its inverse are both nonnegative then it is the permutation of a positive diagonal matrix

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The theorem

- Given a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ that its inverse exists. Let $\mathbf{B} = \mathbf{A}^{-1}$. If $\mathbf{A}$ and $\mathbf{B}$ are both (elementwise) nonnegative, then $\mathbf{A}$ is the permutation of a positive diagonal matrix. Mathematically,

$$\mathbf{A} = \mathbf{D}\mathbf{\Pi},$$

where Π is a permutation matrix, and $\mathbf{D}$ is a diagonal matrix with positive diagonal.

- This document: prove this theorem.
(Note: it will be easier to understand the proof by drawing the matrices.)

- Notation: for a matrix $\mathbf{M}$, let $\mathbf{m}_i$ be the ith column and $\mathbf{m}^i$ be the ith row.

Tools for the proof

- (Assumption A1) $\mathbf{A}$ is nonnegative: $\mathbf{A} \geq 0$.
- (Assumption A2) $\mathbf{B}$ is nonnegative: $\mathbf{B} \geq 0$.
- (Assumption A3) $\mathbf{AB} = \mathbf{I}$.
- (Assumption A4) $\mathbf{A}$ and $\mathbf{B}$ are both full rank.
- (Fact F1) For $a \geq 0, b \geq 0$, if $a + b = 0$, then $a = b = 0$.
- (Fact F2) $\mathbf{I}$ is a diagonal matrix that
  - $I_{ij} = 1$ for all $j = i$.
  - $I_{ij} = 0$ for all $j \neq i$.
- (Fact F3) For $\mathbf{C} = \mathbf{AB}$, then $C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$.

The proof ... 1/4

- (A3) and (F2) give

$$\begin{aligned} (\mathbf{AB})_{ij} &= 1 \quad \text{for all } j = i \\ (\mathbf{AB})_{ij} &= 0 \quad \text{for all } j \neq i \end{aligned} \tag{1}$$

- Apply (F3) on (1) gives

$$\begin{aligned} \sum_{k=1}^{n} A_{ik}B_{kj} &= 1 \quad \text{for all } j = i \\ \sum_{k=1}^{n} A_{ik}B_{kj} &= 0 \quad \text{for all } j \neq i \end{aligned} \tag{2}$$

- Apply (F1) on the second equation in (2), it means

$$A_{ik}B_{kj} = 0 \;\; \forall k, \forall j \neq i. \tag{3}$$

What it means: the product between $\mathbf{a}^i$ and $\mathbf{b}_j$ is zero.

The proof ... 2/4

- Apply (A1) and (A2) on (3), it means we cannot have $A_{ik} > 0$ and $B_{kj} > 0$ at the same time , and we can have either
  - $A_{ik} = B_{kj} = 0$
  - $A_{ik} = 0, B_{kj} > 0$
  - $A_{ik} > 0, B_{kj} = 0$

  Or in other words, we have either both of them are zero, or we have a complementary condition between A_{ik} and B_{kj}: if one of them is zero, the other one is not zero.

- Now by (A4), $\mathbf{A}$ and $\mathbf{B}$ cannot have zero column or zero row. Hence for (3),
  - At least one A_{ik} is non-zero (and the corresponding B_{kj} has to be zero).
  - At least one B_{kj} is non-zero (and the corresponding A_{ik} has to be zero).

  That is

$$\text{Each row of } \mathbf{A} \text{ has at least one non-zero, and} \\ \text{each column of } \mathbf{B} \text{ has at least one non-zero.} \tag{4}$$

The proof ... 3/4

- Now we show that: in ith row of $\mathbf{A}$, if the kth entry is nonzero, i.e., $A_{ik} \geq 0$. Then only the kth entry in the ith column of $\mathbf{B}$ is nonzero, i.e., $B_{ki} \geq 0$, or

$$B_{kj} \geq 0, \ j = i \text{ and } B_{kj} = 0, \ j \neq i. \tag{5}$$

- This argument means

$$\begin{aligned} \text{Each row of } \mathbf{A} \text{ has at most one non-zero, and} \\ \text{each column of } \mathbf{B} \text{ has at most one non-zero, and} \\ \text{the location of the nonzero in } \mathbf{A} \text{ and } \mathbf{B} \text{ are linked.} \end{aligned} \tag{6}$$

- Together (4) + (6) imply $\mathbf{A}$ is the permutation of a positive diagonal matrix, and the proof is finished.

- So the remaining task now is to show (5) is true.

The proof ... 4/4 (In this slide, $j \neq i$.)

- We want to show: if $A_{ik} \neq 0$, then $\begin{cases} B_{kj} = 0 \\ B_{ki} \neq 0 \end{cases}$. To show this, we use proof by contradiction.

- Consider I_{ij}:

$$0 \stackrel{(F2)}{=} I_{ij} \stackrel{(F3)}{=} \sum_k A_{ik} B_{kj} \geq A_{ip} B_{pj} > 0.$$

  Hence (5) is true.

- Explanation of the second last inequality: In (5) we want to show $A_{ik} B_{kj} = 0$ proof by contradiction, so we consider $A_{ip} B_{pj} \neq 0$ for some p. As $\mathbf{A}$ and $\mathbf{B}$ are both nonnegative hence $A_{ip} B_{pj} > 0$.

  And for the $\geq$ sign: summing k nonnegative terms is always larger than or equal to a single term in the sum.

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