$\mathbf{AB}$ is nonnegative if cone$(\mathbf{A}^\top) \subseteq$ cone$^*(\mathbf{B})$

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The statement

Given two matrices : $\mathbf{A} \in \mathbb{R}^{m \times r}$ and $\mathbf{B} \in \mathbb{R}^{r \times n}$, the product $\mathbf{C} = \mathbf{AB}$ is a nonnegative matrix if and only if the conical hull of $\mathbf{A}^\top$ is a subspace of the dual of the conical hull of $\mathbf{B}$. Mathematically,

$$\text{cone}(\mathbf{A}^\top) \subseteq \text{cone}^*(\mathbf{B}).$$

Notes :

- A matrix is called nonnegative if all the elements of that matrix are nonnegative real numbers.
- $\mathbf{A}, \mathbf{B}$ can be any matrix, not necessarily nonnegative.

Preliminaries : definitions, in set form

Definition (Nonnegative combination) A *nonnegative combination* of a set of vectors $\{\mathbf{w}_i\}_{i\in[r]} = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is the linear combination of these vectors in the form

$$\alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2 + \cdots + \alpha_r\mathbf{w}_r = \sum_i^r \alpha_i\mathbf{w}_i,$$

where $\alpha_i \geq 0$ for all $i \in \{1, 2, \ldots, r\} := [r]$.
Other name of nonnegative combination : conical combination.

Definition (Cone) Given a set of vectors $\{\mathbf{w}_i\}_{i\in[r]}$, the *cone* of these vectors is the subspace defined as the set of all possible nonnegative combination of these vectors

$$\text{cone}\Big(\{\mathbf{w}_i\}_{i\in[r]}\Big) = \Big\{\mathbf{v} \,\Big|\, \mathbf{v} = \sum_i^r \alpha_i\mathbf{w}_i, \alpha_i \geq 0\Big\}.$$

Other names of cone : conical hull, positive span.

Preliminaries : definitions, in matrix form

Definition (Nonnegative combination) Given a set of vectors stored in a matrix $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r]$, a *nonnegative combination* of these vectors is the linear combination of these vectors in the form

$$\alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2 + \cdots + \alpha_r \mathbf{w}_r = \mathbf{W}\alpha$$

where $\alpha = [\alpha_1 \, \alpha_2 \, \ldots \alpha_r]^\top$ is a non-negative vector.

Definition (Cone) Given a set of vectors $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r]$, the *cone* of these vectors is the subspace defined as the set of all possible nonnegative combination of the vectors

$$\text{cone}\big(\mathbf{W}\big) = \Big\{ \mathbf{v} \,\Big|\, \mathbf{v} = \mathbf{W}\alpha, \alpha \geq 0 \Big\}.$$

Notation $\mathbf{x} \geq 0$ means all element of $\mathbf{x}$ are nonnegative.

Preliminaries : definitions of dual and the dual of cone

Definition (Dual of a set) Given a set A, the dual of A, denoted as A^*, is the set defined as

$$A^* = \Big\{ \mathbf{b} \,\Big|\, \langle \mathbf{b}, \mathbf{a} \rangle \geq 0, \mathbf{a} \in A \Big\}.$$

Geometric interpretation : A^* is the subspace formed by all vector $\mathbf{b}$ (with the same dimension as $\mathbf{a}$) that forms an acute angle with any $\mathbf{a} \in A$.

Theorem (Dual of a cone) Given a matrix $\mathbf{W}$. The dual of cone$(\mathbf{W})$, denoted as cone$^*(\mathbf{W})$, is the subspace formed by all vector $\mathbf{y}$ such that $\mathbf{W}^\top \mathbf{y}$ is nonnegative. Mathematically,

$$\text{cone}^*(\mathbf{W}) := \Big(\text{cone}(\mathbf{W})\Big)^* = \Big\{ \mathbf{y} \,\Big|\, \mathbf{W}^\top \mathbf{y} \geq 0 \Big\}.$$

We will prove this theorem. The proof itself helps to prove the statement in page 1.

Proving the theorem on the dual of a cone

Given a matrix $\mathbf{W}$, we have

$$\text{cone}^*(\mathbf{W}) := \Big(\text{cone}(\mathbf{W})\Big)^* = \Big\{\mathbf{y} \,\big|\, \mathbf{W}^\top \mathbf{y} \geq 0\Big\}.$$

Proof. We prove it backward by considering the set $\Big\{\mathbf{y} \,\big|\, \mathbf{W}^\top \mathbf{y} \geq 0\Big\}$.

$$\begin{aligned}
\Big\{\mathbf{y} \,\big|\, \mathbf{W}^\top \mathbf{y} \geq 0\Big\} &= \Big\{\mathbf{y} \,\big|\, \langle \mathbf{h}, \mathbf{W}^\top \mathbf{y}\rangle \geq 0, \mathbf{h} \geq 0\Big\} && (1)\\
&= \Big\{\mathbf{y} \,\big|\, \langle \mathbf{x}, \mathbf{y}\rangle \geq 0, \mathbf{x} = \mathbf{W}\mathbf{h}, \mathbf{h} \geq 0\Big\} && (2)\\
&= \Big\{\mathbf{y} \,\big|\, \langle \mathbf{x}, \mathbf{y}\rangle \geq 0, \mathbf{x} \in \text{cone}(\mathbf{W})\Big\} && (3)\\
&= \Big(\text{cone}(\mathbf{W})\Big)^* = \text{cone}^*(\mathbf{W}) \quad \square && (4)
\end{aligned}$$

Explanations.

(1): given $\mathbf{a} \geq 0$, then $\langle \mathbf{a}, \mathbf{b}\rangle \geq 0 \iff \mathbf{b} \geq 0$.

(2): simple algebra, $\langle \mathbf{h}, \mathbf{W}^\top \mathbf{y}\rangle = \mathbf{h}^\top \mathbf{W}^\top \mathbf{y} = (\mathbf{W}\mathbf{h})^\top \mathbf{y}$, then let $\mathbf{x} = \mathbf{W}\mathbf{h}$

(3): definition of cone , here $\mathbf{h}$ is the α in p.4

(4): definition of dual, p.5

Proving the statement

Given matrices $\mathbf{A} \in \mathbb{R}^{m\times r}, \mathbf{B} \in \mathbb{R}^{r\times n}$, $\mathbf{AB} \geq 0$ iff

$$\text{cone}(\mathbf{A}^\top) \subseteq \text{cone}^*(\mathbf{B}).$$

Proof. Consider $\mathbf{AB} \geq 0$ is true.

$$\begin{aligned}
\mathbf{AB} \geq 0 \quad &\iff \quad \mathbf{B}^\top\mathbf{A}^\top \geq 0 && (5)\\
&\iff \quad \text{the vector } \mathbf{B}^\top\mathbf{A}(i,:)^\top \text{ is non-negative for all } i && (6)\\
&\iff \quad \mathbf{A}(i,:)^\top \in \{\mathbf{y} \,|\, \mathbf{B}^\top\mathbf{y} \geq 0\} \ \forall i && (7)\\
&\iff \quad \mathbf{A}(i,:)^\top \in \text{cone}^*(\mathbf{B}) \ \forall i && (8)\\
&\iff \quad \text{cone}(\mathbf{A}^\top) \subseteq \text{cone}^*(\mathbf{B}) \quad \square && (9)
\end{aligned}$$

Explanations.

(5): Take transpose.

(6): $\mathbf{B}^\top\mathbf{A}^\top \geq 0$ means $\mathbf{B}^\top$ times each column of $\mathbf{A}^\top$ is non-negative.

(7,8): definition of dual of cone of $\mathbf{B}$ (p.6)

(9): for all i

Examples

Suppose $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Then for $\mathbf{AB} \geq 0$, the rows of $\mathbf{A}$ need to be inside the set

$$\left\{ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \mid \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \right\} \iff \left\{ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \mid y_1 \geq 0, y_2 \geq 0 \right\}.$$

That is, as long as rows of $\mathbf{A}$ are nonnegative, then $\mathbf{AB}$ is nonegative.

Suppose $\mathbf{B} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$.

Then for $\mathbf{AB} \geq 0$, the rows of $\mathbf{A}$ need to be inside the set

$$\left\{ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \mid \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \right\} \iff \left\{ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \mid y_2 \leq y_1, y_2 \leq -y_1 \right\}.$$

The following examples of $\mathbf{A}$ give $\mathbf{AB} \geq 0$

$$\begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & -2 \\ -2 & -3 \end{bmatrix}.$$

Last page - summary

- The dual of the cone$(\mathbf{W})$ is the subspace formed by all vector $\mathbf{y}$ such that $\mathbf{W}^\top\mathbf{y}$ is non-negative. Mathematically

$$\text{cone}(\mathbf{W}) = \Big\{ \mathbf{y} \,\Big|\, \mathbf{W}^\top \mathbf{y} \geq 0 \Big\}.$$

  Note that $\mathbf{W}$ here can be any matrix

- The matrix product $\mathbf{AB}$ is a non-negative matrix if cone$(\mathbf{A}^\top)$ is a subspace of cone$(\mathbf{B})$. Mathematically

$$\text{cone}(\mathbf{A}^\top) \subseteq \text{cone}^*(\mathbf{B}).$$

  Note that $\mathbf{A}, \mathbf{B}$ here can be any matrix (of appropriate size for forming a product)

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