No scaling ambiguity in symmetric NMF

Problem: given $X \in \mathbb{R}^{n\times n}$, the symmetric NMF: find $H \in \mathbb{R}^{n\times r}$ via solving

$$(P) \quad \min_{H \geq 0} f(H) = \frac{1}{2}\|X - HH^T\|_F^2$$

r is factorization rank, and it is given.

Consider (P) is solved exactly and we have

$$X = HH^T.$$

Now, consider the solution is not unique and we have

$$X = \hat{H}\hat{H}^T.$$

We now show $\hat{H}$ is a permuted version of H.

i.e. $H = P\hat{H}$, where P is a permutation and there is no S s.t. $H = PS\hat{H}$, where $S \in \mathbb{R}^{n\times n}$ is a diagonal matrix.

Proof. Consider the general asymmetric case

$$X = WH^T = (WD)(D^{-1}H^T) = \hat{W}\hat{H}^T,$$

where $D \in \mathbb{R}^{r\times r}$ is full rank, this matrix always exist and it is the cause of non-uniqueness of the solution of NMF.

In Symmetric case, $W=H$ so

$$X = HH^T = (HD)(D^{-1}H) = \hat{H}\hat{H}^T,$$

hence $(HD)^T$ has to be equal to $D^{-1}H^T$ to fulfill the condition $W=H$. The equality $(HD)^T = D^{-1}H^T$ gives $D^T = D^{-1}$, meaning D is orthonormal. Hence for (P), non-uniquenesses ambiguity can only be caused by an orthonormal matrix D. Multiplication by orthonormal matrix preserves length. thus there is no scaling ambiguity in SymNMF.

i.e. for $X = HH^T = HSPP^{-1}S^{-1}H^T = \hat{H}\hat{H}^T$

where P is permutation matrix, S is diagonal scaling matrix, S does not exist here. //

Further explanation why no scaling S

Says $H = \hat{H}D$, now consider the norm of the first column of H and $\hat{H}$

$$\|He_1\|_2 = \|\hat{H}De_1\|_2$$

Suppose $D = S$ (scaling only) and the permutation is already "corrected" (w.l.o.g)

Hence we have

$$\|He_1\|_2 = \|\hat{H}Se_1\|_2$$

$$\Leftrightarrow \|h_1\|_2 = \|\hat{h}_1 s_{11}\|_2$$

$$\Leftrightarrow s_{11} = 1.$$

this is true for all i, hence

$$S = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$ and thus there is no S.