Discrete Fourier Series

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First created: 2013-2014. Last update: 2017-Feb-1

Consider periodic sequence $\widetilde{x}[n]$ with period N :

$$\begin{gathered}\widetilde{x}[n]=\widetilde{x}[n+N]\\ \widetilde{x}[n]=\widetilde{x}[n+2N]\\ \vdots\\ \widetilde{x}[n]=\widetilde{x}[n+rN]\end{gathered}$$

Since period is N , so frequency is $\dfrac{1}{N}$ and thus angular frequency $\dfrac{2\pi}{N}$

The signal $\widetilde{x}[n]$ can be represented by Fourier Series :

$$\widetilde{x}[n]=\begin{cases}\dfrac{1}{N}\sum a_k\sin\dfrac{2\pi}{N}kn+b_k\cos\dfrac{2\pi}{N}kn & \text{(Real Fourier Series)}\\ \\ \dfrac{1}{N}\sum c_k e^{j\frac{2\pi}{N}kn} & \text{(Complex Fourier Series)}\end{cases}$$

Since complex Fourier Series is more compact, thus the following discussion will be focus on Complex Fourier Series

Recall that, c_k is actually the the Fourier Coefficients that can be found by some mean, thus

$$\widetilde{x}[n]=\frac{1}{N}\sum\widetilde{X}[k]e^{j\frac{2\pi}{N}kn}$$

Recall that, a function expressed as a sum of series is actually a infinite series (since this is the condition for the function to converge !). Thus the index of the summation can be written as

$$\widetilde{x}[n]=\frac{1}{N}\sum_{k=1}^{\infty}\widetilde{X}[k]e^{j\frac{2\pi}{N}kn}$$

Expand the summation

$$\widetilde{x}[n]=\frac{1}{N}\widetilde{X}[1]e^{j\frac{2\pi}{N}n}+\frac{1}{N}\widetilde{X}[2]e^{j\frac{2\pi}{N}2n}+\frac{1}{N}\widetilde{X}[3]e^{j\frac{2\pi}{N}3n}+...+\frac{1}{N}\widetilde{X}[l]e^{j\frac{2\pi}{N}ln}+...$$

It can be seen that

$$\begin{gathered}\text{when } k=N\;\; e^{j\frac{2\pi}{N}Nn}=\underbrace{e^{j2\pi n}}_{1}=1\\ \text{when } k=N+1\;\; e^{j\frac{2\pi}{N}(N+1)n}=e^{j\frac{2\pi}{N}n}\\ \vdots\\ \text{when } k=rN+l\;\; e^{j\frac{2\pi}{N}(rN+l)n}=e^{j\frac{2\pi}{N}ln}\end{gathered}$$

And since

That is, the series only require first N term , other term are redunant.

$$\widetilde{x}[n] = \frac{1}{N}\sum_{k=1}^{N} \widetilde{X}[k]e^{j\frac{2\pi}{N}kn}$$

Where the last term

$$e^{j\frac{2\pi}{N}kN} = e^{j2\pi k} = 1 = e^{j2\pi 0}$$

Thus

$$\widetilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \widetilde{X}[k]e^{j\frac{2\pi}{N}kn}$$

Therefore, we now obtain the Fourier Series representation of an periodic signal

Then the problem is : what is the Fourier Coefficient and how to find it

Applying *Orthogonality* trick as the continuous counterpart , convolute with $e^{-j\frac{2\pi}{N}rn}$

$$\widetilde{x}[n] * e^{-j\frac{2\pi}{N}rn} = \left(\frac{1}{N}\sum_{k=0}^{N-1} \widetilde{X}[k]e^{j\frac{2\pi}{N}kn}\right) * e^{-j\frac{2\pi}{N}rn}$$

$$\sum_{n=0}^{N-1} \widetilde{x}[n]e^{-j\frac{2\pi}{N}rn} = \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} \widetilde{X}[k]e^{j\frac{2\pi}{N}kn}\right) e^{-j\frac{2\pi}{N}rn}$$

$$\sum_{n=0}^{N-1} \widetilde{x}[n]e^{-j\frac{2\pi}{N}rn} = \sum_{n=0}^{N-1}\frac{1}{N}\sum_{k=0}^{N-1} \widetilde{X}[k]e^{j\frac{2\pi}{N}(k-r)n}$$

Change the summation order

$$\sum_{n=0}^{N-1} \widetilde{x}[n]e^{-j\frac{2\pi}{N}rn} = \sum_{k=0}^{N-1} \widetilde{X}[k]\left(\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n}\right)$$

Let's consider the bracket

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n}$$

Case 1. When $k - r = mN$, $m \in \mathbb{Z}$

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}mNn} = \frac{1}{N}\sum_{n=0}^{N-1} \underbrace{e^{j2\pi mn}}_{1} = \frac{1}{N}\sum_{n=0}^{N-1} 1 = \frac{1}{N}\left(\underbrace{1+1+...+1}_{N}\right) = 1$$

Case 2. When $k - r \neq mN$, $m \in \mathbb{Z}$

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n} = \frac{1}{N}\left[e^{j\frac{2\pi}{N}(k-r)} + e^{j\frac{2\pi}{N}(k-r)2} + e^{j\frac{2\pi}{N}(k-r)3} + ... + e^{j\frac{2\pi}{N}(k-r)N-1}\right]$$

Notice that the equation above is an Geometric Series

$$\frac{1}{N}\left[\left(e^{j\frac{2\pi}{N}(k-r)}\right) + \left(e^{j\frac{2\pi}{N}(k-r)}\right)^2 + \left(e^{j\frac{2\pi}{N}(k-r)}\right)^3 + ... + \left(e^{j\frac{2\pi}{N}(k-r)}\right)^{N-1}\right] = \frac{1}{N}\frac{1-\overbrace{e^{j\frac{2\pi}{N}(k-r)N}}^{1}}{1-e^{j\frac{2\pi}{N}(k-r)}} = 0$$

Thus

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n} = \begin{cases} 1 & k-r=mN \\ \\ 0 & else \end{cases}$$

Therefore

$$\sum_{n=0}^{N-1} \widetilde{x}[n]e^{-j\frac{2\pi}{N}rn} = \sum_{k=0}^{N-1} \widetilde{X}[k]\left(\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n}\right)$$

$$\sum_{n=0}^{N-1} \widetilde{x}[n]e^{-j\frac{2\pi}{N}rn} = \widetilde{X}[1]\left(\frac{1}{N}\underbrace{\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(1-r)n}}_{0}\right)+\widetilde{X}[2]\left(\frac{1}{N}\underbrace{\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(2-r)n}}_{0}\right)+...+\widetilde{X}[r]\left(\frac{1}{N}\underbrace{\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(r-r)n}}_{N}\right)+...$$

$$\sum_{n=0}^{N-1} \widetilde{x}[n]e^{-j\frac{2\pi}{N}rn} = \widetilde{X}[r]$$

i.e. The Fourier Coefficient can be computed by

$$\widetilde{X}[m] = \sum_{n=0}^{N-1} \widetilde{x}[n]e^{-j\frac{2\pi}{N}mn}$$

Notice that $\widetilde{X}[m]$ is also periodic with period N

$$\widetilde{X}[m+rN] = \sum_{n=0}^{N-1} \widetilde{x}[n]e^{-j\frac{2\pi}{N}(m+rN)n} = \sum_{n=0}^{N-1} \widetilde{x}[n]e^{-j\frac{2\pi}{N}mn}\underbrace{e^{-j2\pi rn}}_{1} = \sum_{n=0}^{N-1} \widetilde{x}[n]e^{-j\frac{2\pi}{N}mn} = \widetilde{X}[m]$$

Thus, the Discrete Fourier Series representation of signal $\widetilde{x}[n]$ is thus

$$\widetilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \widetilde{X}[k]e^{j\frac{2\pi}{N}kn}$$

$$\text{where } \widetilde{X}[m] = \sum_{n=0}^{N-1} \widetilde{x}[n]e^{-j\frac{2\pi}{N}mn}$$

Or express in one equation as

$$\widetilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{n=0}^{N-1} \widetilde{x}[n]e^{-j\frac{2\pi}{N}mn}\right)e^{j\frac{2\pi}{N}kn}$$

And where the Fourier Coefficient $\widetilde{X}[m]$, is the Discrete Fourier Transform

$$DFT\left\{\widetilde{x}[n]\right\} = \widetilde{X}[m] = \sum_{n=0}^{N-1} \widetilde{x}[n]e^{-j\frac{2\pi}{N}mn}$$

$$-END-$$