Finding the Zeros of a Digital Filter

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Introduction

A digital system has the following input-ouput relation

$$y[n] = x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4] + x[n-5] + x[n-6] + x[n-7]$$

Apply $\mathcal{Z}$-Transform

$$Y(z) = X(z) + z^{-1}X(z) + z^{-2}X(z) + z^{-3}X(z) + z^{-4}X(z) + z^{-5}X(z) + z^{-6}X(z) + z^{-7}X(z)$$

Thus the system function is

$$H(z) = \frac{Y(z)}{X(z)} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6} + z^{-7}$$

i.e.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z^1 + 1}{z^7}$$

The system's poles and zeros are by solving

$$\begin{cases} z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z^1 + 1 = 0 & \text{Zeros} \\ z^7 = 0 & \text{Poles} \end{cases}$$

To solve for the pole, it is easy. But to solve for the zeros, it is also easy by using a small trick.

$$\left(z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z^1 + 1\right) = \left(z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z^1 + 1\right)\frac{z-1}{z-1} = \frac{z^8 - 1}{z-1}$$

The resultant equation is easy to solve

$$z^8 - 1 = 0 \iff z = 1^{\frac{1}{8}} = (\exp j2k\pi)^{\frac{1}{8}}_{k=0,1,2,\ldots,7}$$

So

$$z = \exp\left(j\frac{0\pi}{8}\right)\ \exp\left(j\frac{2\pi}{8}\right)\ \exp\left(j\frac{4\pi}{8}\right)\ \exp\left(j\frac{6\pi}{8}\right)\ \exp\left(j\frac{8\pi}{8}\right)\ \exp\left(j\frac{10\pi}{8}\right)\ \exp\left(j\frac{12\pi}{8}\right)\ \exp\left(j\frac{14\pi}{8}\right)$$

For the zeros $\exp\left(j\frac{0}{8}\pi\right) = 1$, it cancel out with the poles 1 , so the zeros of the system are

$$\exp\left(j\frac{2\pi}{8}\right)\ \exp\left(j\frac{4\pi}{8}\right)\ \exp\left(j\frac{6\pi}{8}\right)\ \exp\left(j\frac{8\pi}{8}\right)\ \exp\left(j\frac{10\pi}{8}\right)\ \exp\left(j\frac{12\pi}{8}\right)\ \exp\left(j\frac{14\pi}{8}\right)$$

Using conjugate pairs to make it more compact

$$\exp\left(\pm j\frac{\pi}{7}\right)\ \exp\left(\pm j\frac{3\pi}{7}\right)\ \exp\left(\pm j\frac{5\pi}{7}\right)$$

But the original equation does not have the $z-1$ as the pole, so artificially introduced pole at $z=1$ has to be rejected.

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