Thunder Fast review of some equations of Signal and Systems

December 21, 2012

Euler Equations

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \qquad \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Phasor Additions

$$\sum_{k=0}^{n} A_k \cos(\omega_0 t + \phi_k) = Re\left\{\sum_{k=0}^{n} A_k e^{\omega_0 t + \phi_k}\right\} = Re\left\{e^{\omega_0 t}\sum_{k=0}^{n} A_k e^{\phi_k}\right\} = A\cos(\omega_0 t + \phi)$$

Instantaneous Frequency

$$A\cos(\omega_0 t + \phi) = A\cos\Psi \qquad f_i = \frac{1}{2\pi}\frac{d\Psi}{dt}$$

Spectrum Representation of Signals : Fourier Series

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k) = \sum_{k=-\infty}^{\infty} c_k e^{k\omega_0 t + \phi_k}$$

$$\omega_0 = GCD\left[\omega_k\right] \qquad \forall k \qquad T_0 = \frac{2\pi}{\omega_0}$$

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{k\omega_0 t} dt$$

Fourier Series

If $x(t)$ has Fourier Series Expansion with coefficient c_k , and $y(t) = A(t)x(t)$, then $y(t)$ has Fourier Series Expansion with coefficient d_k where

$$d_k = A(t)c_k$$

Shannon-Nyquist Sampling Theorem

$$x[n] = x(nT_s) \qquad T_s = \frac{1}{f_s}$$

$$f_s \geq 2f_{Max}$$

Normalized Frequency

$$x[n] = x(nT_s) = A\cos(\omega_0 nT_s + \phi) = A\cos(\hat{\omega}n + \phi) \qquad \hat{\omega} = \frac{\omega_0}{f_s} = 2\pi\frac{f_0}{f_s} \text{ dimensionless}$$

Signal Reconstruction using Pulse Interpolation

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]\text{Pul}\,(t - nT_s)$$

$$\text{Ideal Pul}\,(t) = \text{sinc}\left(\frac{\pi}{T_s}t\right) = \frac{\sin\left(\frac{\pi}{T_s}t\right)}{\frac{\pi}{T_s}t}$$

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]\frac{\sin\left(\frac{\pi}{T_s}t - n\pi\right)}{\frac{\pi}{T_s}t - n\pi}$$

General FIR Filter

$$y[n] = \sum_{k=0}^{M} b_n x[n-k]$$

- The M is order of FIR filter
- If b_k is same $\forall k$, it is a causal running average of $M+1$ sample
- If b_k is not constant, it is a weighted running average of $M+1$ sample
- In MATLAB, b = [...] ; where M =length(b) - 1

LTI System

- Additive : for input $x[n] = x_1[n] + x_2[n]$, the output is $y[n] = y_1[n] + y_2[n]$
- Homogenous : for input $x[n] = \alpha x_1[n]$, the output is $y[n] = \alpha y_1[n]$
- Additive + Homogenous = Linear
- Shift invariance : for input $x[n-n_0]$, output is $y[n-n_0]$

Convolution

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Frequency Response

With complex exponential input $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, the output is

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \quad \Longrightarrow \quad y[n] = \left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}\right) Ae^{j\phi}e^{j\hat{\omega}n}$$

$$H(\hat{\omega}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} = |H(\hat{\omega})| \angle H(\hat{\omega})$$

Conjugate Symmetry of Frequency Response

$$|H(\hat{\omega})| = |H(-\hat{\omega})| \qquad \Re e\left[H(-\hat{\omega})\right] = \Re e\left[H(\hat{\omega})\right]$$

$$\angle H(-\hat{\omega}) = -\angle H(\hat{\omega}) \qquad \Im m\left[H(-\hat{\omega})\right] = -\Im m\left[H(\hat{\omega})\right]$$

Frequency Response of L-point Average

$$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k]$$

$$H(\hat{\omega}) = \frac{1}{L}\sum_{k=0}^{L-1} e^{-j\hat{\omega}k} = \frac{1}{L}\frac{1-e^{-j\hat{\omega}L}}{1-e^{-j\hat{\omega}}} = \frac{1}{L}\frac{e^{-j\hat{\omega}L/2}\left(e^{j\hat{\omega}L/2}-e^{-j\hat{\omega}L/2}\right)}{e^{-j\hat{\omega}/2}\left(e^{j\hat{\omega}/2}-e^{-j\hat{\omega}/2}\right)} = \left(\frac{1}{L}\frac{\sin\frac{\hat{\omega}L}{2}}{\sin\frac{\hat{\omega}}{2}}\right)e^{-j\hat{\omega}(L-1)/2}$$

Frequency Response of Difference Equation

$$y[n] = x[n] - x[n-k]$$

$$H(\hat{\omega}) = 1 - e^{-j\hat{\omega}k} = e^{-j\hat{\omega}k/2}\left[e^{j\hat{\omega}k/2} - e^{-j\hat{\omega}k/2}\right] = 2\sin\frac{\hat{\omega}k}{2}e^{j\left(\frac{\pi}{2}-\frac{k\hat{\omega}}{2}\right)}$$

$$|H(\hat{\omega})| = 2\left|\sin\frac{\hat{\omega}k}{2}\right| \qquad \angle H(\hat{\omega})\begin{cases} \frac{\pi}{2}-\frac{\hat{\omega}}{2} & 0<\hat{\omega}<\pi \\ -\pi+\frac{\pi}{2}-\frac{\hat{\omega}}{2} & -\pi<\hat{\omega}<0 \end{cases}$$

Using Z-Transfrom to find Convolution

$$h[n] * x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} H(z)X(z)$$

L-point running average in Z-domain

$$H(z) = \sum_{k=0}^{L-1} z^{-k} = \frac{1-z^{-L}}{1-z^{-1}} = \frac{z^L - 1}{z^{L-1}(z-1)}$$

$$z^L - 1 = 0 \quad \Rightarrow \quad z^L = 1 \quad z = e^{j\frac{2\pi k}{L}}_{k=0,1,\ldots,L-1}$$

$$H(z) = \sum_{k=0}^{L-1} z^{-k} = \prod_{k=1}^{L-1}\left(1 - e^{j\frac{2\pi k}{L}}z^{-1}\right)$$

There are $L-1$ poles in zero, $L-1$ zeros on the unit circle. The pole in $1+0j$ cancel with the zero there.

Bandpass filter

$$H(z) = \prod_{\substack{k=0 \\ k \neq k_0}}^{L-1} \left(1 - e^{j\frac{2\pi k}{L}} z^{-1}\right)$$

IIR Filters

$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k]$$

General First Order and Second Order IIR Filter

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$H_I(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$H_{II}(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

IIR Stability

Causal LTI IIR system is stable if all poles and zero lies strictly inside unit circle in z-plane

Continuous Time system

- Input/output

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

- Stable

$$y(t) = \int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

- Causal

$$h(\tau) = 0 \quad \forall \tau < 0$$

- Frequency Response

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt = H(s)|_{s=j\omega} = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

- Sinusoidal input/output

$$y(t) = |H(\omega_0)| A \cos(\omega_0 t + \phi + \angle H(\omega_0))$$

Ideal Systems

- Delay

$$H(\omega) = e^{-j\omega t_d}$$

- Lowpass

$$H(\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

- Highpass

$$H(\omega) = \begin{cases} 0 & |\omega| \le \omega_c \\ 1 & |\omega| > \omega_c \end{cases}$$

- Bandpass

$$H(\omega) = \begin{cases} 1 & \omega_{c1} \le |\omega| \le \omega_{c2} \\ 0 & \text{else} \end{cases}$$

$-END-$