The Sinc Dirichlet Function

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$$Diric\,(x) = \frac{\sin\dfrac{nx}{2}}{n\sin\frac{x}{2}}$$

1 The L-point running-average filter

$$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k]$$

The output is the average of $x[n]$ and the previous $L-1$ samples (total L samples)

$$y[n] = \frac{1}{L}\,(\,x[n] + x[n-1] + x[n-2] + ... + x[n-L+2] + x[n-L+1]\,)$$

This is an example of linear time-invariant system

Consider the frequency response
i.e. When input is $x[n] = Ae^{j(n\hat{\omega}+\phi)}$

$$y(\hat{\omega}) = \frac{1}{L}\sum_{k=0}^{L-1} Ae^{j((n-k)\hat{\omega}+\phi)} = \left(\frac{1}{L}\sum_{k=0}^{L-1} e^{-j\hat{\omega}k}\right) Ae^{j(n\hat{\omega}+\phi)}$$

$$H\,(\hat{\omega}) = \frac{1}{L}\sum_{k=0}^{L-1} e^{-j\hat{\omega}k}$$

Apply the sum of geometric series

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$$

Thus

$$H\,(\hat{\omega}) = \frac{1}{L}\sum_{k=0}^{L-1} e^{-j\hat{\omega}k}$$

$$= \frac{1}{L}\frac{1-e^{-j\hat{\omega}L}}{1-e^{-j\hat{\omega}}}$$

Apply a trick

$$1-e^{j\hat{\omega}N} = e^{j\hat{\omega}N/2}\left(e^{-j\hat{\omega}N/2} - e^{j\hat{\omega}N/2}\right) \qquad 1-e^{-j\hat{\omega}N} = e^{-j\hat{\omega}N/2}\left(e^{j\hat{\omega}N/2} - e^{-j\hat{\omega}N/2}\right)$$

$$H(\hat{\omega}) = \frac{1}{L}\frac{e^{-j\hat{\omega}L/2}\left(e^{j\hat{\omega}L/2} - e^{-j\hat{\omega}L/2}\right)}{e^{-j\hat{\omega}/2}\left(e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2)}\right)}$$

$$= \frac{1}{L}\frac{\left(e^{j\hat{\omega}L/2} - e^{-j\hat{\omega}L/2}\right)\cdot\dfrac{1}{2j}}{\left(e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2)}\right)\cdot\frac{1}{2j}}\cdot\frac{e^{-j\hat{\omega}L/2}}{e^{-j\hat{\omega}/2}}$$

$$= \frac{1}{L}\frac{\sin\frac{\hat{\omega}L}{2}}{\sin\frac{\hat{\omega}}{2}}e^{-j\hat{\omega}(L-1)/2}$$

$$H(\hat{\omega}) = Diric(\hat{\omega})\, e^{-j\hat{\omega}(L-1)/2}$$

Thus, the Diric function is the magnitude component of the L-point running average filter

2 The Dirichlet Function

$$Diric(x) = \frac{\sin\dfrac{nx}{2}}{n\sin\dfrac{x}{2}}$$

When $x = 2k\pi \quad k \in \mathbb{Z}$

$$Diric(x) = \frac{\sin nk\pi}{n\sin k\pi}$$

$$= \frac{\sin\dfrac{nx}{2}}{n\dfrac{x}{2}}\frac{\dfrac{x}{2}}{\sin\dfrac{x}{2}} = \frac{\operatorname{sinc}\dfrac{nx}{2}}{\operatorname{sinc}\dfrac{x}{2}}$$

Therefore

$$\lim_{x\to 0} Diric(x) = \lim_{x\to 0}\frac{\operatorname{sinc}\dfrac{nx}{2}}{\operatorname{sinc}\dfrac{x}{2}}$$

Since the limit exists for both upper side and lower side

$$= \frac{\lim\limits_{x\to 0}\operatorname{sinc}\dfrac{nx}{2}}{\lim\limits_{x\to 0}\operatorname{sinc}\dfrac{x}{2}} = \frac{1}{1} = 1$$

Thus

$$\lim_{x\to 0} Diric(x) = 1$$

Also, it can be shown that

$$\max\ Diric(x) = 1$$

$$-END-$$