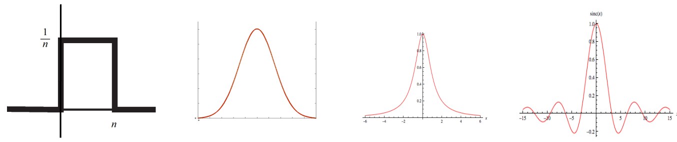
# The Dirac Delta Function(al) $\delta(t)$



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September 9, 2013

The purpose of this document is to **illustrate the properties of Dirac Delta Function** $\int_{-\infty}^{\infty} \delta(x)dx = 1$

## 1 Definition

The Dirac Delta Function is *defined by its assigned properties*

1. It dacays

$$\delta(x) = 0 \ \ , x \neq 0$$

2. Screening property

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

Where $f(x)$ is well-defined ordinary function

By property 2, if $f(x)$ is the unity function , i.e. $f(x) = 1(t) = 1$

$$\int_{-\infty}^{\infty} 1(x)\delta(x)dx = 1(0) = 1$$

Thus

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

Therefore, the Dirac Delta function has the following properties

It is infinitely thin spike
It is infinitely high
It is not ordinary function

## 2 Dirac Delta Function as a Functional

Since there is no ordinary function has such properties, the Dirac Delta function has to be represented as a *limit* of certain function

That means, Dirac Delta Function has *several expressions / several forms*. And all these forms must filfill the assigned properties (1) and (2)

In this way, the Dirac Delta Function is actually *a function of functions* called *generalized function* or *functional* or *distribution*

It is a $functionof\ functions$ because it describe the general behaviours of a set of functions.

Following are some candidates of Dirac Delta Function

$$\delta_n(t) = \begin{cases} \frac{1}{n} & 0 \le x \le n \\ 0 & \text{else} \end{cases} \quad \delta_n(t) = \begin{cases} n & |x| \le \frac{1}{2n} \\ 0 & \text{else} \end{cases} \quad \delta_n(t) = \sqrt{\frac{n^2}{\pi}} \exp\left(-n^2x^2\right) \quad \delta_n(t) = \frac{n}{\pi}\frac{1}{1+n^2x^2} \quad \delta_n(t) = \frac{\sin nx}{\pi x}$$

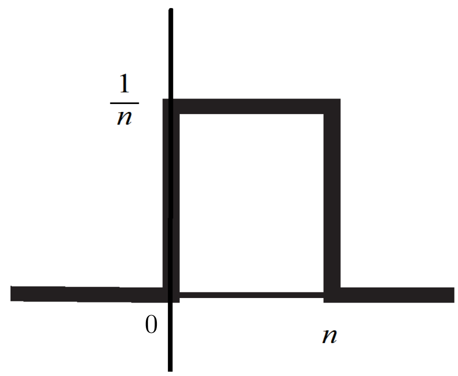
With these ordinary function representation, the Dirac Delta Function can be represented as

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = \lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\delta_n(x)dx = f(0)$$

### 2.1 Rectangular Pulse

$$\delta_n(t) = \begin{cases} \frac{1}{n} & 0 \le x \le n \\ 0 & \text{else} \end{cases}$$

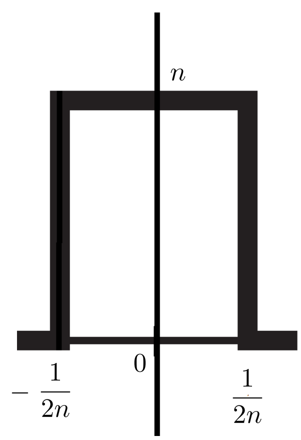
$$\begin{aligned} & \textstyle\int_{-\infty}^{\infty} \delta(x)dx \\ = & \lim_{n\to\infty} \textstyle\int_{-\infty}^{\infty} \delta_n(x)dx \\ = & \lim_{n\to\infty} \left[ \textstyle\int_{-\infty}^{0} \underbrace{\delta_n(x)}_{0} dx + \int_{0}^{n} \delta_n(x)dx + \int_{n}^{\infty} \underbrace{\delta_n(x)}_{0} dx \right] \\ = & \lim_{n\to\infty} \textstyle\int_{0}^{n} \frac{1}{n}dx \\ = & \lim_{n\to\infty} 1 \\ = & 1 \end{aligned}$$




### 2.2 Central Rectangular pulse

$$\delta_n(t) = \begin{cases} n & |x| \le \frac{1}{2n} \\ 0 & \text{else} \end{cases}$$

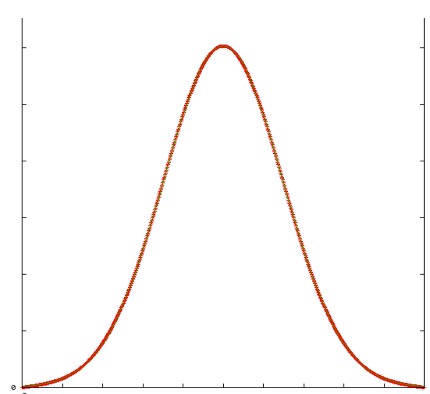
$$\begin{aligned} & \textstyle\int_{-\infty}^{\infty} \delta(x)dx \\ = & \lim_{n\to\infty} \left( \textstyle\int_{-\infty}^{-1/2n} + \int_{-1/2n}^{1/2n} + \int_{1/2n}^{\infty} \right) \delta_n(x)dx \\ = & \lim_{n\to\infty} n \textstyle\int_{-1/2n}^{1/2n} dx \\ = & \lim_{n\to\infty} n \left[ \frac{1}{2n} - \left( \frac{-1}{2n} \right) \right] \\ = & \lim_{n\to\infty} 1 \\ = & 1 \end{aligned}$$




### 2.3 Gaussian Pulse

$$\delta_n(t) = \sqrt{\frac{n^2}{\pi}} \exp\left(-n^2x^2\right)$$

$$\begin{aligned} & \textstyle\int_{-\infty}^{\infty} \delta(x)dx \\ = & \lim_{n\to\infty} \textstyle\int_{-\infty}^{\infty} \sqrt{\frac{n^2}{\pi}} \exp\left(-n^2x^2\right) dx \\ = & \lim_{n\to\infty} \sqrt{\frac{1}{\pi}} \textstyle\int_{-\infty}^{\infty} \exp\left(-n^2x^2\right) dnx \\ = & \lim_{n\to\infty} \sqrt{\frac{1}{\pi}} \textstyle\int_{-\infty}^{\infty} \exp\left(-y^2\right) dy \\ = & \lim_{n\to\infty} 2\sqrt{\frac{1}{\pi}} \textstyle\int_{0}^{\infty} \exp\left(-y^2\right) dy \end{aligned}$$



Using Jacobian Transformation technique

$$\int_0^{\infty} \exp\left(-y^2\right) dy = \frac{\sqrt{\pi}}{2}$$

**The Jacobian Transformation Technique** ( For reference only ! )

The following is an illustration of the Jacobian Transformation technique

$$\begin{aligned} & \int_0^\infty e^{-y^2} dy \\ = & \sqrt{\left(\int_0^\infty e^{-y^2} dy\right)\left(\int_0^\infty e^{-y^2} dy\right)} \\ = & \sqrt{\left(\int_0^\infty e^{-y^2} dy\right)\left(\int_0^\infty e^{-x^2} dx\right)} \\ = & \sqrt{\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy} \end{aligned}$$

From Rectangular $(x, y)$ →Polar $(r, \theta)$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2 = r^2 \\ \theta = \tan^{-1}\frac{y}{x} \end{cases} \qquad \begin{cases} x \in [0,\infty) \\ y \in [0,\infty) \end{cases} \rightarrow \begin{cases} r \in [0,\infty) \\ \theta \in [0, \frac{\pi}{2}] \end{cases}$$

$$dxdy = Jdrd\theta$$

Where the Jacobian

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det\begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} = r$$

Thus

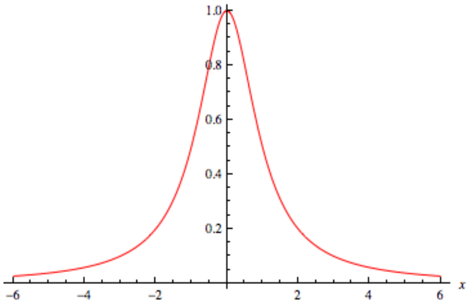
$$\begin{aligned} & \int_0^\infty e^{-y^2} dy \\ = & \sqrt{\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy} \\ = & \sqrt{\int_0^{\pi/2} \int_0^\infty e^{-r^2} rdrd\theta} \\ = & \sqrt{\int_0^{\pi/2} d\theta \int_0^\infty e^{-r^2} rdr} \\ = & \sqrt{\frac{\pi}{2}\int_0^\infty e^{-r^2}\frac{1}{2}dr^2} \\ = & \sqrt{\frac{\pi}{4}\int_0^\infty e^{-r^2} dr^2} \\ = & \sqrt{\frac{\pi}{4}} \end{aligned}$$

Therefore

$$\begin{aligned} & \int_{-\infty}^\infty \delta(x)dx \\ = & \lim_{n\to\infty} \int_{-\infty}^\infty \sqrt{\frac{n^2}{\pi}} \exp\left(-n^2x^2\right) dx \\ = & \lim_{n\to\infty} 2\sqrt{\frac{1}{\pi}} \int_0^\infty \exp\left(-y^2\right) dy \\ = & \lim_{n\to\infty} 2\sqrt{\frac{1}{\pi}}\sqrt{\frac{\pi}{4}} \\ = & \lim_{n\to\infty} 1 \\ = & 1 \end{aligned}$$

## 2.4 Lorentzian Pulse

$$\delta_n(t) = \frac{n}{\pi}\frac{1}{1+n^2x^2}$$




$$\begin{aligned} & \int_{-\infty}^{\infty}\delta(x)dx \\ = & \lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{n}{\pi}\frac{1}{1+n^2x^2}dx \\ = & \lim_{n\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+n^2x^2}dnx \\ = & \lim_{u\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+u^2}du \\ = & \lim_{u\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+u^2}du \end{aligned}$$

$$\int\frac{1}{1+u^2}du = \tan^{-1}u$$

( For reference only , but actually you should already know this !!! )

The following illustrate how to calculate this integral

Let $u = \tan\theta$ , $du = \sec^2\theta d\theta$ , at $u = \infty$, $\theta = +\frac{\pi}{2}$ , at $u = -\infty$, $\theta = -\frac{\pi}{2}$

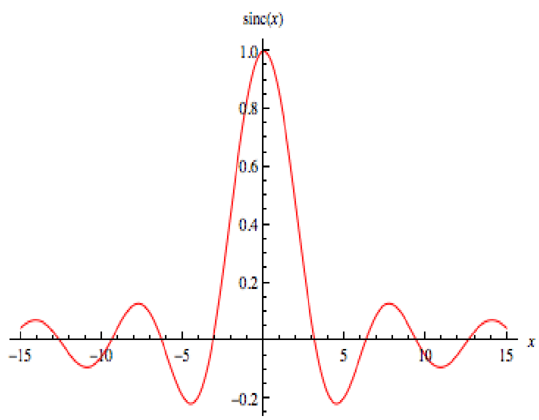
$$\begin{aligned} & \int\frac{1}{1+u^2}du \\ = & \int\frac{1}{1+\tan^2\theta}\sec^2\theta d\theta \\ = & \int d\theta \\ = & \theta \\ = & \tan^{-1}u \end{aligned}$$

Thus

$$\begin{aligned} & \lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{n}{\pi}\frac{1}{1+n^2x^2}dx \\ = & \frac{1}{\pi}\lim_{u\to\infty}\int_{0}^{\infty}\frac{1}{1+u^2}du \\ = & \frac{1}{\pi}\lim_{u\to\infty}\left[\tan^{-1}u\right]_{-\infty}^{\infty} \\ = & \frac{1}{\pi}\left[\frac{\pi}{2}-\left(\frac{-\pi}{2}\right)\right] \\ = & 1 \end{aligned}$$

## 2.5 Sinc Pulse

$$\delta_n(t) = \frac{\sin nx}{\pi x}$$




$$\begin{aligned} & \int_{-\infty}^{\infty}\delta(x)dx \\ = & \lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{\sin nx}{\pi x}dx \\ = & \lim_{n\to\infty}\frac{n}{\pi}\int_{-\infty}^{\infty}\frac{\sin nx}{nx}dx \\ = & \lim_{n\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{sinc} nx dnx \\ = & \lim_{u\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{sinc} u du \end{aligned}$$

The integral $\int_{-\infty}^{\infty}\mathrm{sinc}udu$ , the Bilateral Sinc integral , $\mathrm{BiSi}(\infty)$ , has the value of

$$\int_{-\infty}^{\infty}\mathrm{sinc}udu = \pi$$

( All for reference , but actually you should already know this !!!!! )

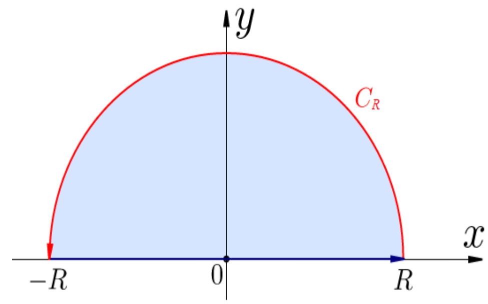
The following illustrate how to do this integral using Jordan Contour Integral Technique in Complex Analysis

Consider

$$\frac{e^{jz}}{z} \quad \text{has a simple pole at } z = 0$$

To use the Jordan method, construct a function with removable singularity at $z = 0$

$$f(z) = \frac{e^{jz} - 1}{z}$$




Consider Cauchy's integral theorem

$$\oint f(z)dz = \underbrace{\int_{-R}^{R} f(x)dx}_{\text{along x-axis}} + \underbrace{\int_{C} f(z)dz}_{\text{circular contour}} = 0$$

So

$$\begin{aligned}
& \oint f(z)dz \\
= & \underbrace{\int_{-R}^{R} f(x)dx}_{\text{along x-axis}} + \underbrace{\int_{C} f(z)dz}_{\text{circular contour}} \\
= & \textstyle\int_{-R}^{R} f(x)dx + \int_C \frac{e^{jz}-1}{z}dz \\
= & \textstyle\int_{-R}^{R} f(x)dx + \int_C \frac{-1}{z}dz + \int_C \frac{e^{jz}}{z}dz
\end{aligned}$$

Now consider the $\int_C \frac{1}{z}dz - \int_C \frac{e^{jz}}{z}dz$

For the first integral

$$\begin{aligned}
& \textstyle\int_C \frac{1}{z}dz \\
= & \textstyle\int_0^{\pi} e^{-j\theta} j e^{j\theta} d\theta \quad (z = e^{j\theta}) \\
= & \textstyle\int_0^{\pi} j d\theta \\
= & j\pi
\end{aligned}$$

For the second integral, consider the ML-inequality

$$\left| \int_C \frac{e^{jz}}{z} dz \right| \le \frac{1}{R} \int_C |e^{jz}||dz| < \frac{1}{R} \int_0^{\pi} dz = \frac{\pi}{R}$$

Therefore

$$\oint f(z)dz = \int_{-R}^{R} f(x)dx - j\pi + \int_C \frac{e^{jz}}{z}dz = 0$$

$$\Longleftrightarrow \int_{-R}^{R} f(x)dx - j\pi + \int_C \frac{e^{jz}}{z}dz = 0$$

$$\Longleftrightarrow \int_{-R}^{R} f(x)dx - j\pi = -\int_C \frac{e^{jz}}{z}dz$$

$$\Longleftrightarrow \left| \int_{-R}^{R} f(x)dx - j\pi \right| = \left| \int_C \frac{e^{jz}}{z}dz \right| < \frac{\pi}{R}$$

$$\Longleftrightarrow \left| \int_{-R}^{R} f(x)dx - j\pi \right| < \frac{\pi}{R}$$

( All for reference , but actually you should already know this !!!!! )

Now consider

$$\begin{aligned}\text{Im}\left[\int_{-R}^{R} f(z)dz\right] &= Im\int_{-R}^{R}\frac{e^{jz}-1}{z}dz\\ &= m\int_{-R}^{R}\frac{\cos z + j\sin z - 1}{z}dz\\ &= \int_{-R}^{R}\frac{\sin z}{z}dz\\ &= \int_{-R}^{R} f(z)dz\end{aligned}$$

Therefore

$$\left|\int_{-R}^{R} f(x)dx - j\pi\right| < \frac{\pi}{R}$$

$$\Longleftrightarrow \left|Im\int_{-R}^{R} f(z)dz - j\pi\right| < \frac{\pi}{R}$$

Thus the imaginary unit can be taken out

$$\Longleftrightarrow \left|\int_{-R}^{R} f(x)dx - \pi\right| < \frac{\pi}{R}$$

Notice that since absolute value is non-negative

$$0 < \left|\int_{-R}^{R}\frac{\sin x}{x}dx - \pi\right| < \frac{\pi}{R}$$

Now , to show the middle term is zero, we have to take the limit

$$\lim_{R\to\infty}\frac{\pi}{R} = 0$$

Thus, by Squeeze Theorem

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{\sin x}{x}dx - \pi = 0$$

$$\Longleftrightarrow \lim_{R\to\infty}\int_{-R}^{R}\frac{\sin x}{x}dx = \int_{-\infty}^{\infty}\frac{\sin x}{x}dx = \pi$$

And therefore

$$\begin{aligned}&\int_{-\infty}^{\infty}\delta(x)dx\\ =& \lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{\sin nx}{\pi x}dx\\ =& \lim_{u\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\text{sinc}u\,du\\ =& \lim_{n\to\infty}\frac{1}{\pi}\pi\\ =& 1\end{aligned}$$

$-END-$