Magnitude response of $\dfrac{1}{1-az^{-1}}$

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Suppose we now have a transfer function $H(z) = \dfrac{1}{1-az^{-1}}$, what we want is to calculate the magnitude frequency respond $|H(\hat{\omega})|$

First we put $z = e^{j\hat{\omega}}$ to obtain the frequency response

$$H(\hat{\omega}) = \frac{1}{1-ae^{-j\hat{\omega}}}$$

Next , inorder to ompute $|H(\hat{\omega})|$, instead of considering $|H\left(\hat{\omega}\right)|$ directly, we consider $|H(\hat{\omega})|^2$ instead.

Also, instead of computing $\left|H\left(\hat{\omega}\right)\right|^2 = \left|H\left(\hat{\omega}\right)\right|\left|H\left(\hat{\omega}\right)\right|$, we compute $\left|H\left(\hat{\omega}\right)\right|^2 = \left|H\left(\hat{\omega}\right)\right|\left|H\left(-\hat{\omega}\right)\right|$

$$\begin{aligned}
|H(\hat{\omega})|^2 &= \left|\frac{1}{1-ae^{-j\hat{\omega}}}\cdot\frac{1}{1-ae^{j\hat{\omega}}}\right| \\
&= \left|\frac{1}{1-ae^{j\hat{\omega}}-ae^{-j\hat{\omega}}+a^2}\right| \\
&= \left|\frac{1}{1-a(e^{j\omega}+e^{-j\omega})+a^2}\right| \\
&= \left|\frac{1}{1-2a\cos\hat{\omega}+a^2}\right|
\end{aligned}$$

And therefore

$$\left|H\left(\hat{\omega}\right)\right| = \sqrt{\left|\frac{1}{1-2a\cos\hat{\omega}+a^2}\right|}$$

Therefore, for example

$$H(z) = \frac{1}{1-0.9z^{-1}}$$

Then the frequency respond is

$$H(\hat{\omega}) = \frac{1}{1 - 0.9e^{-j\hat{\omega}}}$$

And therefore

$$|H(\hat{\omega})|^2 = \left| \frac{1}{1 - 2(0.9)\cos\hat{\omega} + 0.9^2} \right|$$

$$|H(\hat{\omega})|^2 = \left| \frac{1}{1.81 - 1.8\cos\hat{\omega}} \right|$$

Since $|\cos\hat{\omega}| \leq 1$, so $1.81 - 1.8\cos\hat{\omega}$ will always be positive, thus the value of $\hat{\omega}$ will not affect the sign of the denominator, so we can remove the absolute sign.

$$|H(\omega)|^2 = \frac{1}{1.81 - 1.8\cos\hat{\omega}}$$

Hence the $|H(\hat{\omega})|$ is

$$|H(\hat{\omega})| = \frac{1}{\sqrt{1.81 - 1.8\cos\hat{\omega}}}$$