MEDE2500 Tutorial 2
2016-Oct

Content 1. Periodic Functions
2. Spectrum
3. Fourier Series

1. Periodic Functions

A function (or signal) *f*(t) is periodic if *f*(t) = *f*(t+T), where T is a constant, and called *period*. It is important to note that, the constant T should be the smallest possible T.

EXAMPLE 1. Find the period of $\cos\left(\frac{t}{3}\right)+\cos\left(\frac{t}{4}\right)$.
Since cosine is periodic, so by *f*(t) = *f*(t+T), we can write down

$$\cos\left(\frac{t}{3}\right)+\cos\left(\frac{t}{4}\right)=\cos\left(\frac{t+T_1}{3}\right)+\cos\left(\frac{t+T_2}{4}\right) \quad \text{......... (1)}$$

And since *we know that* $\cos(\theta) = \cos(\theta+2m\pi)$, where *m* is integer. Therefore

$$\cos\left(\frac{t}{3}\right)+\cos\left(\frac{t}{4}\right)=\cos\left(\frac{t}{3}+2m_1\pi\right)+\cos\left(\frac{t}{4}+2m_2\pi\right) \quad \text{.......... (2)}$$

(1) and (2) are equal, thus

$$2m_1\pi=\frac{T_1}{3} \text{ and } 2m_2\pi=\frac{T_2}{4}$$

Since $\cos\left(\frac{t}{3}\right)+\cos\left(\frac{t}{4}\right)$ is now a single function, and thus the components $\cos\left(\frac{t}{3}\right), \cos\left(\frac{t}{4}\right)$ should have common period, thus $T_1=T_2$.

Hence

$$2m_1\pi=\frac{T}{3} \text{ and } 2m_2\pi=\frac{T}{4}$$

Or

$$6m_1\pi=T \text{ and } 8m_2\pi=T$$

The least common multiple (LCM) of 6 and 8 is 24

So the smallest T is 24π and $m_1=4$ and $m_2=3$. (Meaning that $\cos\left(\frac{t}{3}\right), \cos\left(\frac{t}{4}\right)$ will see each other after $\cos\left(\frac{t}{3}\right)$ has 4 oscillation and $\cos\left(\frac{t}{4}\right)$ has 3 oscillation.)

EXAMPLE 2. Find the period of $\cos(\omega_1 t) + \cos(\omega_2 t)$.

By *f*(t) = *f*(t+T),

$$\cos(\omega_1 t) + \cos(\omega_2 t) = \cos(\omega_1 t + \omega_1 T) + \cos(\omega_2 t + \omega_2 T)\ .$$

By $\cos(\theta)\ = \cos(\theta + 2m\pi)$,

$$\cos(\omega_1 t + 2m_1\pi) + \cos(\omega_2 t + 2m_2\pi) = \cos(\omega_1 t + \omega_1 T) + \cos(\omega_2 t + \omega_2 T)\ .$$

Hence

$$2m_1\pi = \omega_1 T \text{ and } 2m_2\pi = \omega_2 T$$

Divide the first equation by the second equation

$$\frac{2m_1\pi}{2m_2\pi} = \frac{\omega_1 T}{\omega_2 T}$$

Thus

$$\frac{m_1}{m_2} = \frac{\omega_1}{\omega_2} \ \ldots\ldots\ (3)$$

Meaning that, if $\frac{\omega_1}{\omega_2}$ is a rational number, then it is possible for the two wave to see each other after certain number of oscillation. And in this case $\cos(\omega_1 t) + \cos(\omega_2 t)$ is periodic!

EXAMPLE 3. Use (3) in example 2 to check are the following functions periodic.
(a) $\cos(8t + 0.4\pi) + 3\cos(11t)$
(b) $\cos(2\pi 4t) + \cos(3\pi 6t)$
(c) $\cos(\pi t) + \sin(0.5t)$
(d) $\sin\left(\sqrt{2}t\right) - \sin(t)$
(e) $\sin\left(\sqrt{2}t\right) - \sin\left((\sqrt{2} + \pi)t\right)$

Answer:

(a) Periodic, period is 88π.
(b) Periodic, period is 36.
(c) Aperiodic
(d) Aperiodic
(e) Aperiodic

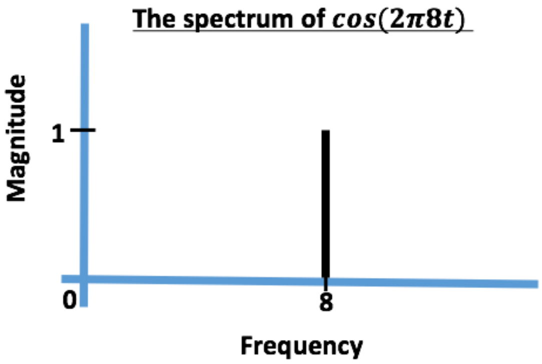
2. Spectrum

A function (signal) in time domain can be expressed as a *spectrum* in frequency domain by a *frequency transform*. If a function is periodic. The spectrum will be a discrete bar-plot. If a function is aperiodic, the spectrum will be a continuous curve.

EXAMPLE 4. Plot the spectrum of $\cos(2\pi 8t)$.

We all know that a sin-cos wave has the general form $A\cos(2\pi ft + \phi)$, where A is the magnitude, f is frequency in Hz, t is time and ϕ is phase shift.

The signal $\cos(2\pi 8t)$ has frequency 8Hz. So the spectrum is

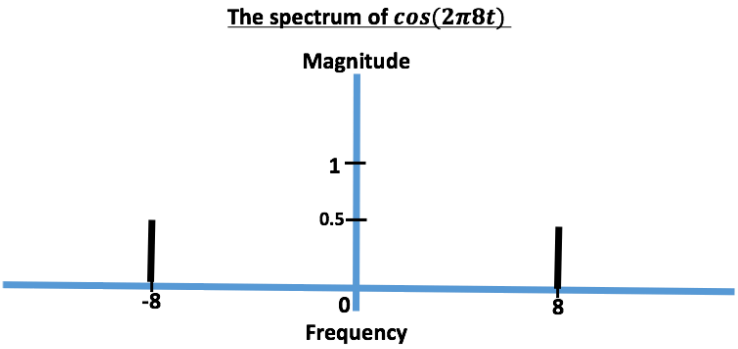


However, *the complex spectrum* is always preferred over the real-spectrum.

By Euler's formula

$$\cos(2\pi 8t) = \frac{\exp(j2\pi 8t) + \exp(-j2\pi 8t)}{2}$$
$$= \frac{1}{2}e^{j2\pi 8t} + \frac{1}{2}e^{-j2\pi 8t}$$

The complex spectrum is



The reason why complex-spectrum is better than real-spectrum is due to the fact that complex-spectrum can also express the phase information in the spectrum plot.

EXAMPLE 5. Plot the spectrum of $2.5\sin(2\pi 2t + 0.3\pi) + 3\cos(2\pi 8t)$.

By Euler's formula

$$2.5\sin(2\pi 2t + 0.3\pi) = 2.5\frac{\exp(j(2\pi 2t + 0.3\pi)) - \exp(-j(2\pi 2t + 0.3\pi))}{2j}$$
$$= \frac{2.5}{2j}e^{j0.3\pi}e^{j2\pi 2t} - \frac{2.5}{2j}e^{-j0.3\pi}e^{-j2\pi 2t}$$

Since $j = e^{j0.5\pi}$ and $-1 = e^{j\pi}$

$$= 1.25e^{-j0.2\pi}e^{j2\pi 2t} + 1.25e^{j0.2\pi}e^{-j2\pi 2t}$$

The final spectrum is

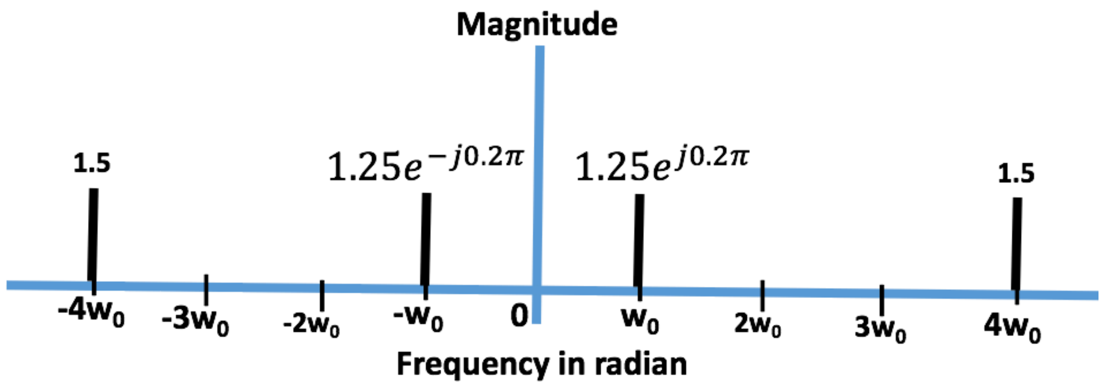
$$2.5\sin(2\pi 2t + 0.3\pi) + 3\cos(2\pi 8t) = 1.25e^{-j0.2\pi}e^{j2\pi 2t} + 1.25e^{j0.2\pi}e^{-j2\pi 2t} + 1.5e^{j2\pi 8t} + 1.5e^{-j2\pi 8t}$$

With a little notation change, let $f_0 = 2$. Then $8 = 4f_0$. The spectrum becomes

$$2.5\sin(2\pi 2t + 0.3\pi) + 3\cos(2\pi 8t) = 1.25e^{-j0.2\pi}e^{j2\pi f_0 t} + 1.25e^{j0.2\pi}e^{-j2\pi f_0 t} + 1.5e^{j2\pi 4 f_0 t} + 1.5e^{-j2\pi 4 f_0 t}$$

Further denote that $\omega_0 = 2\pi f_0$, the expression becomes

$$2.5\sin(2\pi 2t + 0.3\pi) + 3\cos(2\pi 8t) = 1.25e^{-j0.2\pi}e^{j\omega_0 t} + 1.25e^{j0.2\pi}e^{-j\omega_0 t} + 1.5e^{j4\omega_0 t} + 1.5e^{-j4\omega_0 t}$$



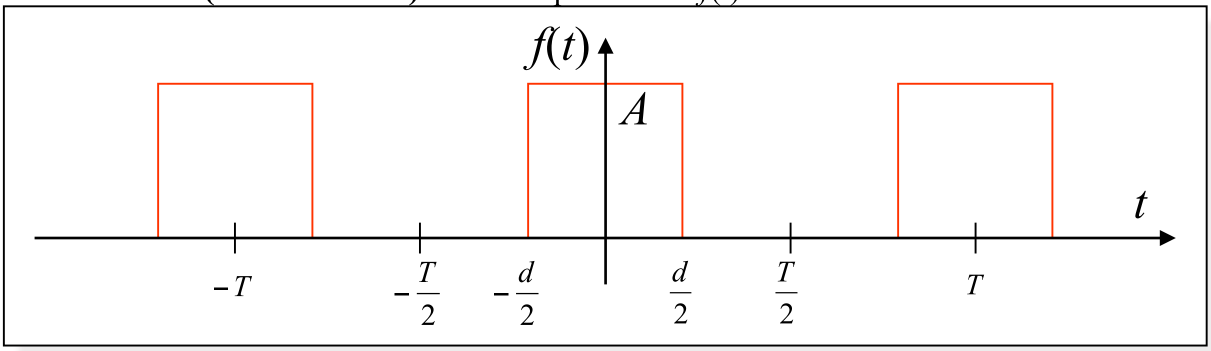
From the example, we see that for a periodic function *f*(t), it can be expressed in a form as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}.$$

Which is called *The (Complex) Fourier Series*. The value near the bar in the plot above are the c_n in the Fourier Series. From this example, $c_1 = 1.25e^{j0.2\pi}$ and $c_{-1} = 1.25e^{-j0.2\pi}$, $c_4 = 1.5$, $c_{-4} = 1.5$. And $c_0 = c_2 = c_3 = 0$ since there is nothing in $0\omega_0, 2\omega_0, 3\omega_0$.

So what is *n* ? *n* is the harmonic number.
When *n* = 1, the component $c_1 e^{j1\omega_0 t}$ is called *Fundamental* harmonics.
When *n* =≥ 2, the components are called overtone.
When *n* = 0, the component $c_0 e^{j0\omega_0 t} = c_0$ is the DC component.

EXAMPLE 6. **(Fourier Series)** Find the spectrum of *f*(t)



To find the spectrum, what we need is to find the unknowns c_n from the (Complex) Fourier Series $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$. The formula to compute c_n is

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt$$

The signal has amplitude *A* for time [-$d/2$ $d/2$] and zero elsewhere, so

$$c_n = \frac{1}{T}\int_{-2/2}^{d/2} Ae^{-jn\omega_0 t}dt$$

Perform the integration

$$\begin{aligned} c_n &= \frac{A}{T}\frac{1}{-jn\omega_0}e^{-jn\omega_0 t}\Big|_{-d/2}^{d/2} \\ &= \frac{A}{T}\frac{1}{-jn\omega_0}\left(e^{-jn\omega_0 d/2} - e^{jn\omega_0 d/2}\right) \end{aligned}$$

Use Euler Formula for sin

$$\begin{aligned} c_n &= \frac{A}{T}\frac{1}{n\omega_0}\left(2\sin\frac{n\omega_0 d}{2}\right) \\ &= \frac{A}{T}\frac{\sin\frac{n\omega_0 d}{2}}{\frac{1}{2}n\omega_0} \\ &= \frac{Ad}{T}\frac{\sin\frac{n\omega_0 d}{2}}{\frac{n\omega_0 d}{2}} \end{aligned}$$

The function $\frac{\sin x}{x}$ is called $\operatorname{sinc} x$

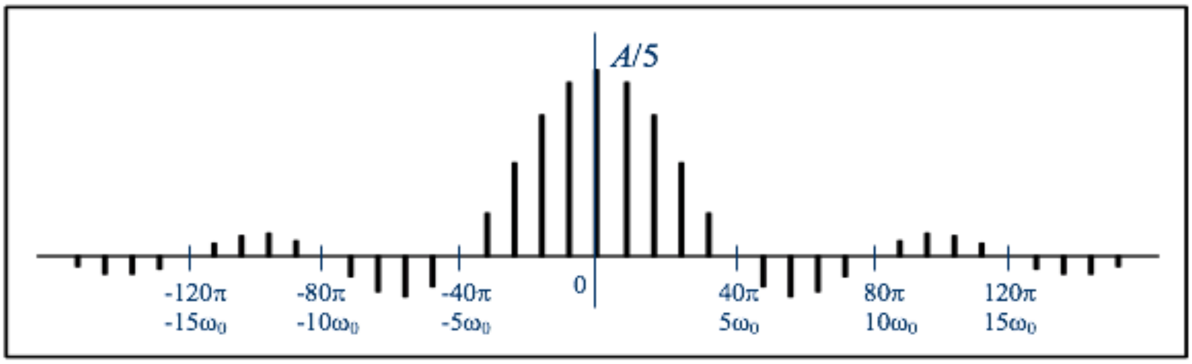
$$c_n = \frac{Ad}{T}\operatorname{sinc}\frac{n\omega_0 d}{2}$$

EXAMPLE. 7. Consider example 6 with different T.

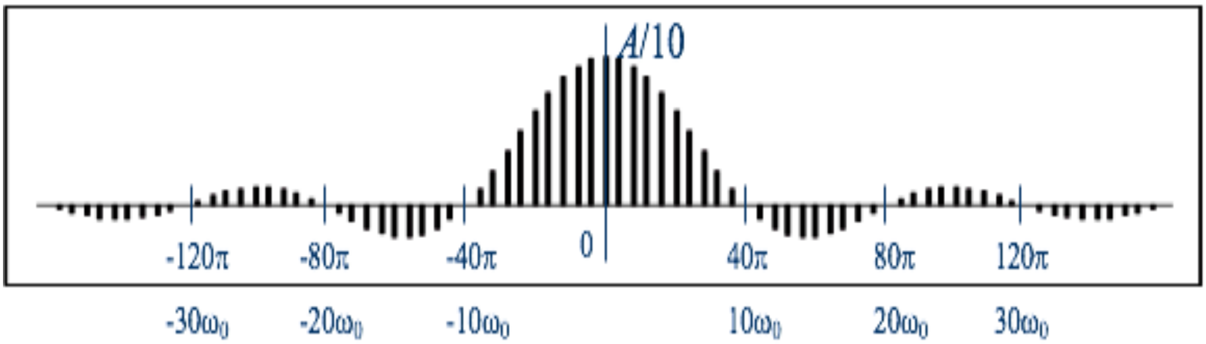
The expression of the coefficient is

$$c_n = \frac{Ad}{T}\operatorname{sinc}\frac{n\omega_0 d}{2}$$

Consider if $d = \frac{1}{20}$, $T = \frac{1}{4}$, then $\omega_0 = \frac{2\pi}{T} = 8\pi$ and $c_n = \frac{A}{5}\operatorname{sinc}\frac{n\omega_0}{40}$. The spectrum is



Consider other parameters keep the same, but $T = \frac{1}{2}$ (longer period), then $\omega_0 = \frac{2\pi}{T} = 4\pi$ and $c_n = \frac{A}{10}\operatorname{sinc}\frac{n\omega_0}{40}$. The spectrum is



One can see that the effect of increasing T reduces the gap between the bars.

Consider other parameters keep the same, but $T \to \infty$ (becomes aperiodic signal). The spectrum above becomes a continuous curve. (Gap between the bars becomes zero).

Notice that in case example, increasing T does not means reducing frequency!!! For sinusoidal signal, $f = \frac{1}{T}$ means if T increase *f* decrease. But for the signal in example 6 and 7, they are not sinusoid but rectangular signal!!! Indeed, the frequency of this signal is so complicated that cannot be described using one number, but a function (the spectrum!). And the frequency (spectrum) of this signal did change when T increase, but it is not as simple as "T increase *f* decrease". And instead, the behavior illustrated in the two figures above!

Reference. *Fourier Series*, T.W. Lu.

-END of Document-

MEDE2500 Tutorial 3
2016-Nov-7

Content 1. The Dirac Delta Function, singularity functions, even and odd functions
2. The sampling process and aliasing
3. A simple filtering system

1a. Dirac Delta Function

The following comes from chapter 1 of my other document on Delta function. (Other parts of the pdf is related to integration on delta function, which is not the main concern here http://www.eee.hku.hk/~msang/DiracDeltaFunction_Ang.pdf)

1 Definition

The Dirac Delta Function is *defined by its assigned properties*

1. It dacays

$$\delta(x) = 0 \ , x \neq 0$$

2. Screening property

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

Where $f(x)$ is well-defined ordinary function

By property 2, if $f(x)$ is the unity function , i.e. $f(x) = 1(t) = 1$

$$\int_{-\infty}^{\infty} 1(x)\delta(x)dx = 1(0) = 1$$

Thus

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

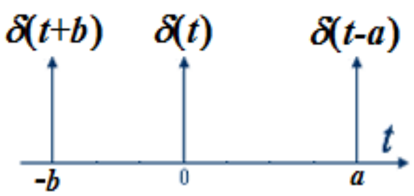
Therefore, the Dirac Delta function has the following properties

It is infinitely thin spike
It is infinitely high
It is not ordinary function

For simplicity, we can say that the *Unit Delta Function* has the form

$$\delta(t) = \begin{cases} 1 & if\ t = 0 \\ 0 & else \end{cases}$$

The following shows some delta functions with different shift



The other name of Delta function is *impulse function*, *Dirac pulse function*.

Impulse is very useful. It can be used to generate other function.

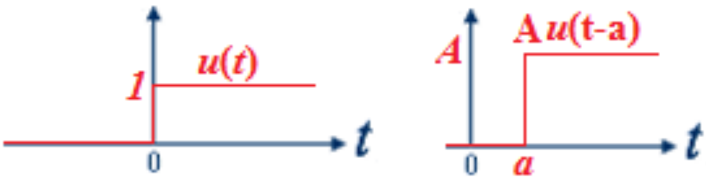
The unit step function (Heaviside Unit Step function)

$$u(t) = \int_{-\infty}^{t} \delta(s)ds$$

For simplicity, we can say that the *Unit Step Function* has the form

$$u(t) = \begin{cases} 1 & if\ t \geq 0 \\ 0 & else \end{cases}$$

The following shows some step functions

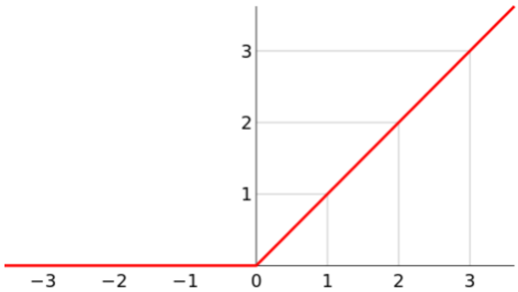


The ramp function

$$r(t) = \int_{-\infty}^{t} u(s)ds$$

For simplicity, we can say that the *ramp function* has the form

$$r(t) = \begin{cases} t & if\ t \geq 0 \\ 0 & else \end{cases}$$



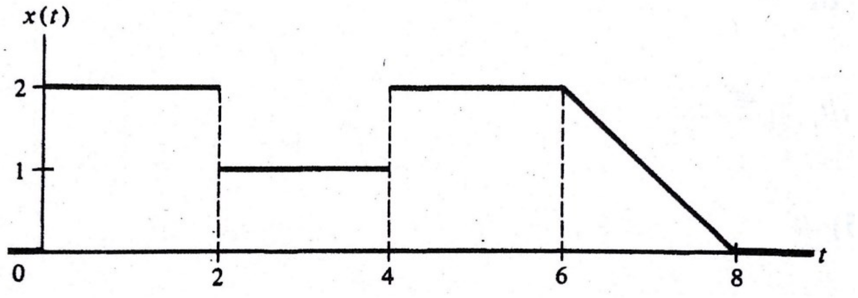
Relationship between impulse, step and ramp

$\delta\ = \frac{d}{dt}u = \frac{d^2}{dt^2}r\ ,\ \ u = \frac{d}{dt}r = \int \delta ds\ \ \ , r = \int uds = \int\int \delta ds^2$

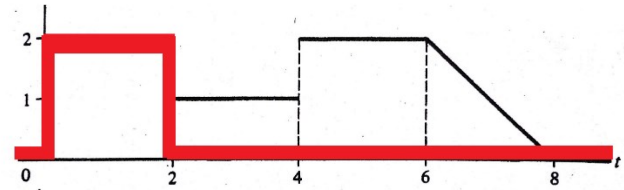
(δ, u, r) together are called *singular functions*

Singular function are useful for expressing other function.

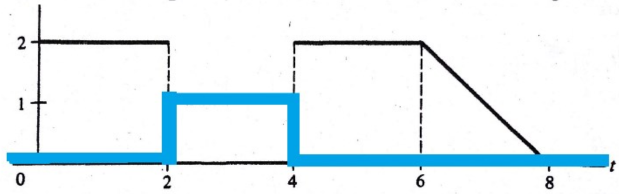
EXAMPLE 1. Write down the expression of the following signal.



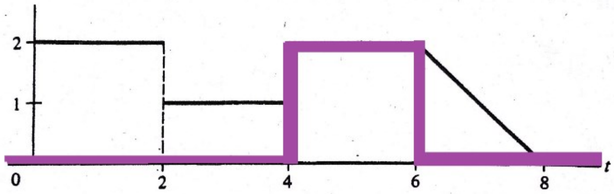
Answer



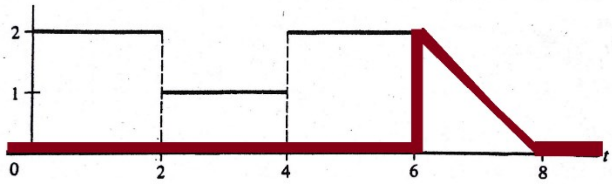
2[u(t) - u(t-2)]



u(t-2) - u(t-4)



2[u(t-4) - u(t-6)]



2[u(t-6) - u(t-8)] [1 - r(t-6)]

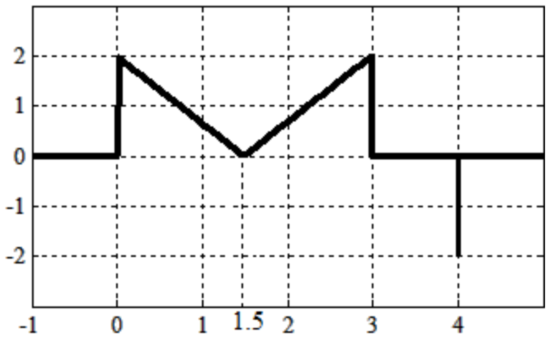
Note.

$$x(t) = 2(u(t) - u(t-2)) + (u(t-2) - u(t-4)) + 2(u(t-4) - u(t-6)) \\ + 2(u(t-6) - u(t-8))(1 - r(t-6))$$

After simplification

$$x(t) = 2u(t) - u(t-2) + u(t-4) - 2u(t-8) - 2(u(t-6) - u(t-8))r(t-6)$$

EXAMPLE 2. Write down the expression of the following signal.



Answer:

$$[u(t)-u(t-1.5)]\frac{2}{1.5}[1-r(t)]$$
$$+[u(t-1.5)-u(t-3)]\frac{2}{1.5}r(t-1.5)$$
$$-2\delta(t-4)$$

EXAMPLE 3. Write down the expression of the following signal. (Called *impulse train* or *Daric comb*).

Answer: $s(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$

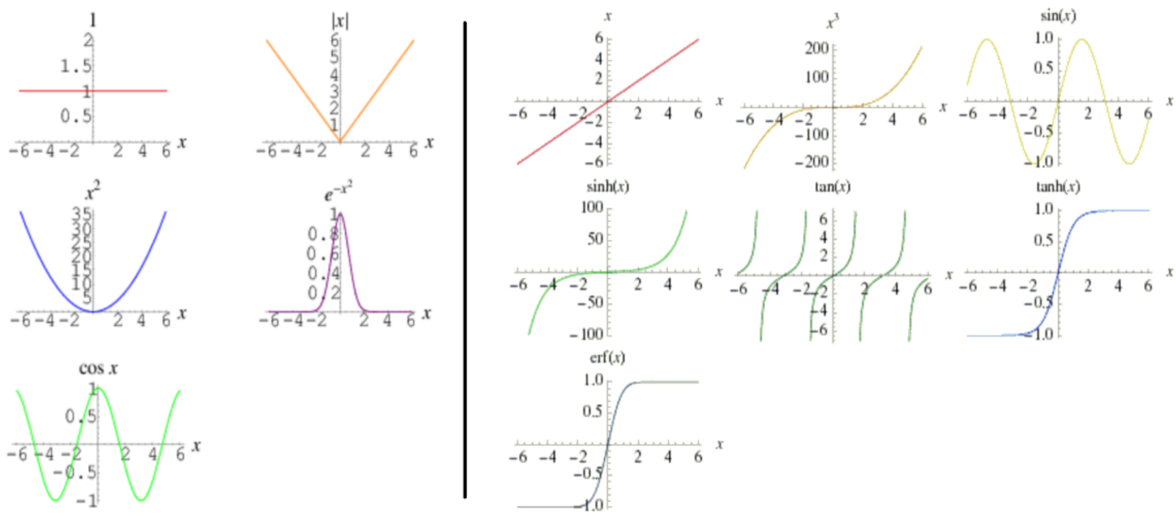
1b. Even and Odd function

A function is even if it is symmetric along the *y*-axis. A function is even if it is anti-symmetrical along the *y*-axis.

$$f \text{ is even} \leftrightarrow f(x)=f(-x)$$
$$f \text{ is odd} \leftrightarrow f(x)=-f(-x)$$

Example of even and odd function

EVEN **ODD**



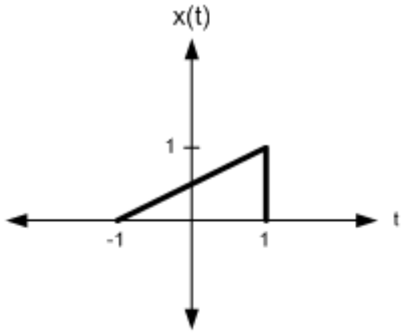
Even and odd function decomposition theorem. Any function can be written as a sum of even and odd functions.

$$f(x) = f_e(x) + f_o(x)$$

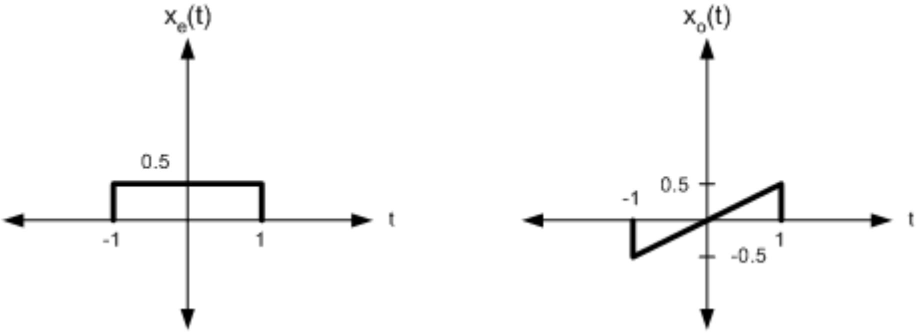
where

$$f_e(x) = \frac{1}{2}\big(f(x) + f(-x)\big)$$
$$f_o(x) = \frac{1}{2}\big(f(x) - f(-x)\big)$$

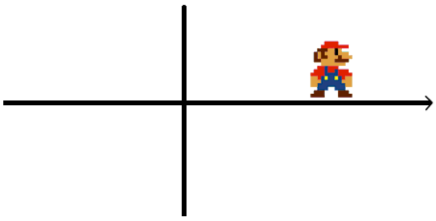
Decomposition of function as even and odd function.
EXAMPLE 4. Decompose the function below into even and odd function.



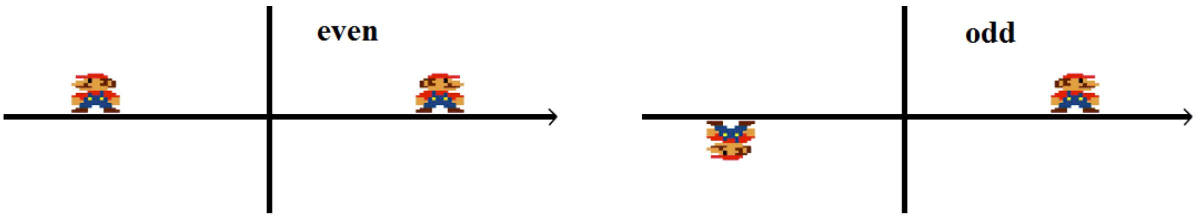
Answer:



EXAMPLE 5. Decompose the function below into even and odd function.



Answer:



Why care about even and odd function: speed up calculation.
EXAMPLE 6. Find the spectrum of $x(t) = \frac{1}{2}[u(t-1) - u(t+1)]$. (The x_e in example 4).

The standard way:

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \frac{1}{2}\int_{-1}^{1} e^{-j\omega t}dt \\
&= \frac{-1}{2j\omega}\, e^{-j\omega t}|_{-1}^{1} \\
&= \frac{-1}{2j\omega}\left(e^{-j\omega} - e^{j\omega}\right) \\
&= \frac{1}{\omega}\sin\omega \\
&= \operatorname{sinc}\omega
\end{aligned}
$$

The “fast” way:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Expand the $e^{-j\omega t}$ using Euler formula

$$X(\omega) = \int_{-\infty}^{\infty} x(t)[\cos\omega t - j\sin\omega t]dt$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\cos\omega t\, dt - j\int_{-\infty}^{\infty} x(t)\sin\omega t\, dt$$

Note that (i) $x(t)$ is even and $\sin\omega t$ is odd, (ii) $even \times odd = odd$. (iii) the integration of odd function is zero. Thus we can ignore the second term.

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)\cos\omega t\, dt \\
&= \frac{1}{2}\int_{-1}^{1} \cos\omega t\, dt \\
&= \frac{1}{2\omega}\sin\omega t\,|_{-1}^{1} \\
&= \frac{1}{\omega}\sin\omega \\
&= \operatorname{sinc}\omega
\end{aligned}
$$

EXAMPLE 7. Find the spectrum of $x_0(t)$ in example 4.

$$x_o(t) = \frac{t}{2}[u(t-1) - u(t+1)]$$

"Normal way"

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$= \frac{1}{2}\int_{-1}^{1} te^{-j\omega t}dt$$

$$= \frac{1}{-2j\omega}\left[t e^{-j\omega t}\Big|_{-1}^{1} - \int_{-1}^{1} e^{-j\omega t}dt \right]$$

$$= \frac{-1}{2j\omega}\left[\left[e^{-j\omega} + e^{+j\omega}\right] - \frac{1}{-j\omega}\left[e^{-j\omega t}\right]_{-1}^{1} \right]$$

$$= \frac{-1}{2j\omega}\left[e^{-j\omega} + e^{j\omega} + \frac{1}{j\omega}\left(e^{-j\omega} - e^{+j\omega}\right) \right]$$

$$= \frac{-1}{j\omega}\left[\cos\omega + \frac{-\sin\omega}{\omega} \right]$$

$$= \frac{1}{j}\,\frac{\sin\omega - \omega\cos\omega}{\omega^2}$$

"Fast way"

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Expand the exp term

$$= \int_{-\infty}^{\infty} x(t)[\cos\omega t - j\sin\omega t]dt$$

Can ignore the first term since it is zero.

$$X(\omega) = \frac{-j}{2}\int_{-1}^{1} t\sin\omega t\, dt$$

Perform integration by parts

$$= \frac{j}{2\omega}\int_{-1}^{1} t d cos\omega t$$

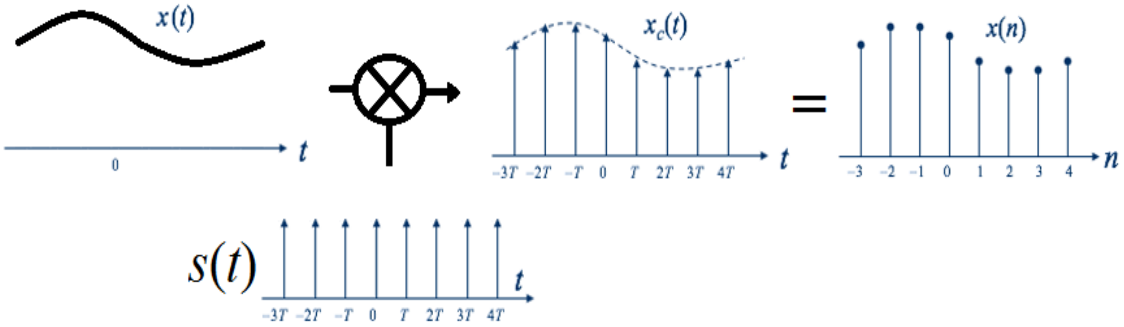
$$= \frac{j}{2\omega}\left[2\cos\omega - \frac{2}{\omega}\sin\omega\right]$$

$$= -j\frac{\sin\omega - \omega\cos\omega}{\omega^2}$$

Message from this example: both are clumsy. But making use of even-odd function may help reduce some of the step!!!

2a. Sampling

Sampling turns a continuous time signal *x*(*t*) into discrete time signal *x*[*n*]. There are many sampling schemes but usually *uniform sampling* will be used: the sampling interval is fix (constant *T*). Discrete signal is obtained by multiplying a sampling signal *s*(*t*) with the original function.



s(*t*) is an impulse train

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

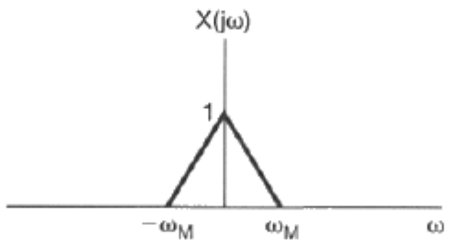
Therefore, the multiplication between $x(t)$ and $s(t)$ creates a signal $x(nT)$

$$\begin{aligned} x_c(t) &= x(nT) \\ &= x(t)s(t) \\ &= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \end{aligned}$$

Now, consider the spectrum. Recall that

$$\text{Spectrum of a signal } v(t) = \begin{cases} \text{Fourier Series of } v(t) \text{ if } v(t) \text{is periodic} \\ \text{Fourier Transform of } v(t) \text{ if } v(t) \text{is aperiodic} \end{cases}$$

Let's assume the spectrum of *x*(*t*) is $X(\omega)$. (or $X(j\omega)$ as shown below.)



The spectrum is *band-limited*. That means the frequency content (complex spectrum) of the signal *x*(*t*) is *within* a frequency range $[-\omega_M \quad +\omega_M]$ (subscript M means max).

In "drawing" a spectrum, actually any kind of shape is possible. But usually triangle will be used. A triangle spectrum means "strong low frequency, weak high frequency".

Now assumes we have the spectrum $S(\omega)$ of $s(t)$. To find the spectrum $X_c(\omega)$ of $x_c(t)$, there is a theorem:

Convolution theorem on spectrum. The relationship between $X(\omega)$, $S(\omega)$ and $X_c(\omega)$ is

$$X_c(\omega) = X(\omega) * S(\omega)$$

where * is convolution.

<u>Some technical details</u>

1. The complete theorem

(i). Two signal multiplied together in the time domain is equivalent to the two signal convoluted together in the frequency domain.

(ii). Two signal convoluted in the time domain is equivalent to the two signal multiplied together in the frequency domain.

i.e.

$$x(t) * s(t) \leftrightarrow X(\omega)S(\omega)$$

$$x(t)s(t) \leftrightarrow X(\omega) * S(\omega)$$

2. Normalization constant

Sometime there will be a normalization constant before the expression. For example

$$x(t)s(t) \leftrightarrow \frac{1}{2\pi}X(\omega) * S(\omega) \qquad or\ \ x(t) * s(t) \leftrightarrow \frac{1}{\sqrt{2\pi}}X(\omega)S(\omega)$$

These are just different notations used in different books.

The spectrum - the Fourier Series of $s(t)$.

Since $s(t)$ is periodic. So we compute the Fourier series.

Fourier series of $s(t)$ is

$$s(t) = \sum_{m=-\infty}^{\infty} c_m e^{jm\omega t}$$

where

$$c_m = \frac{1}{T}\int_{-T/2}^{T/2} s(t)e^{jm\omega t}dt$$

Plug in the equation of $s(t)$ into the integration

$$c_m = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} \delta(t-nT)\, e^{jm\omega t}dt$$

Since integral range is $[-\frac{T}{2}\ \ +\frac{T}{2}]$ and $\delta(t-nT) = 0$ for $t \neq nT$, so all the terms except $\delta(t)$ in the summation vanish.

$$c_m = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{jm\omega t}dt$$

Mathematics trick: since $\delta(t) = 0$ for $t \neq 0$, we can extend the integration range

$$c_m = \frac{1}{T}\int_{-\infty}^{\infty} \delta(t)e^{jm\omega t}dt$$

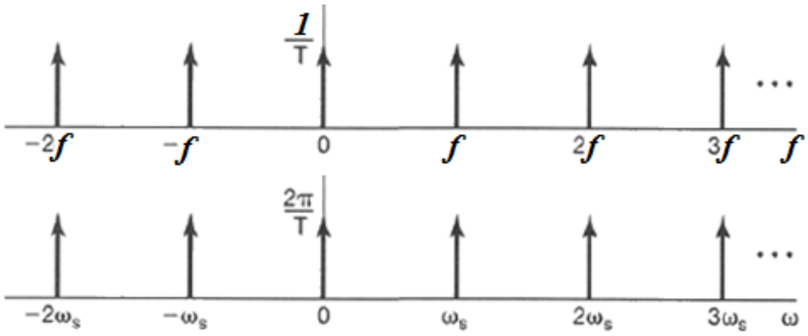
Now we can use the "screening property" of Delta function (page 1 property 2)

$$c_n = \frac{1}{T}e^{jm\omega 0}$$
$$= \frac{1}{T}$$

Thus the Fourier series of $s(t)$ is

$$s(t) = \frac{1}{T}\sum_{m=-\infty}^{\infty} e^{jm\omega t}$$

Now, recall that spectrum of a periodic signal are the coefficient of the Fourier series (tutorial 2). So the spectrum is



As the spectrum is also an impulse train, so it can be expressed mathematically as

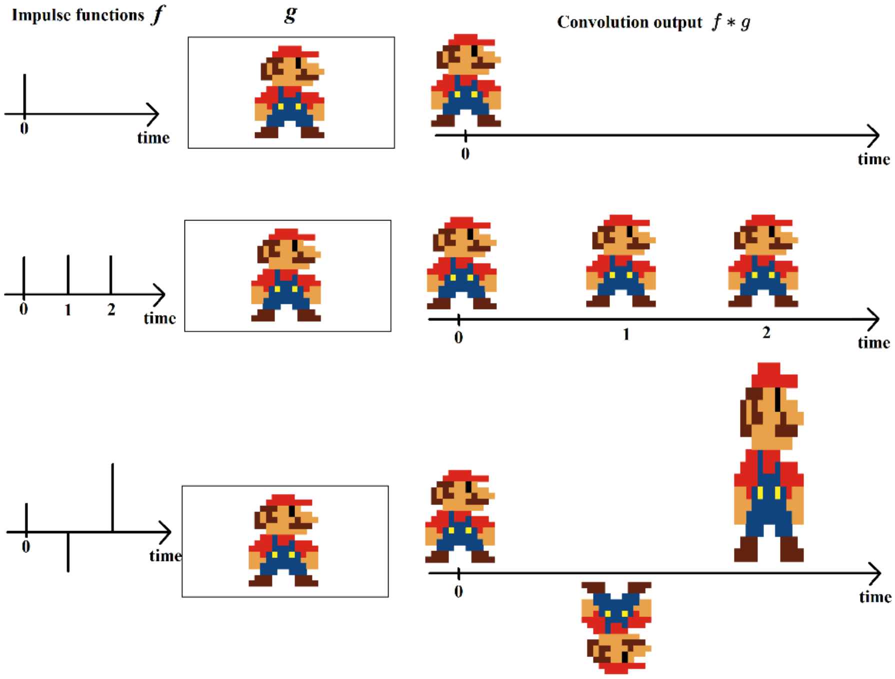
$$S(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \quad or \quad S(\omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega)$$

Note. If $s(t)$ is not *infinite* pulse train but *finite* pulse train. Then it is NOT periodic function and the process above changed from Fourier series to Fourier Transform.

What we have now:
(i) All the expressions of $x(t), s(t)$, $x_c(t), X(\omega)$ and $S(\omega)$
(ii) The relationships between them. (The convolution theorem on spectrum).

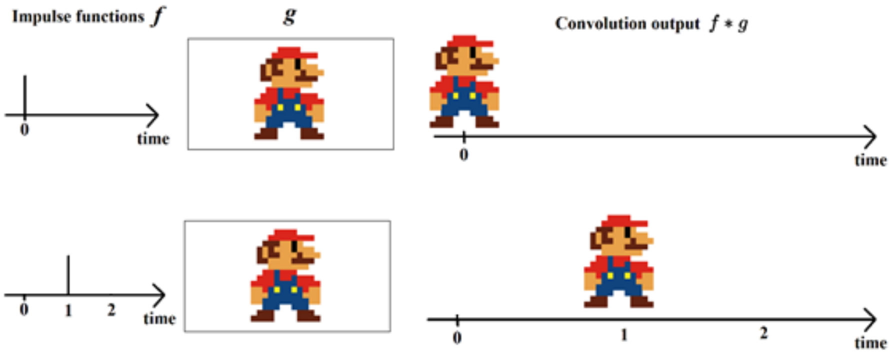
From (ii), what we need to do is to convolute $X(\omega)$ and $S(\omega)$ to get the spectrum of the sampled signal.

Convolution with impulse train illustration

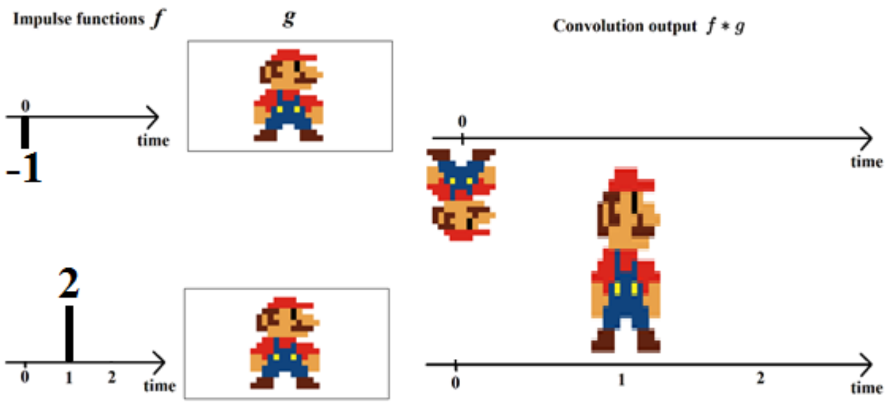


How to "think" about convolution

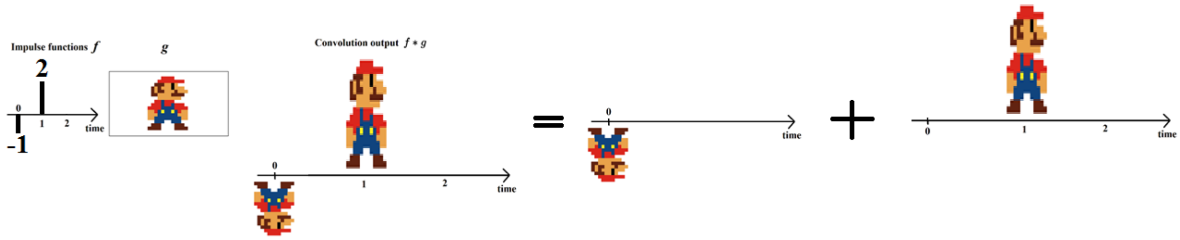
"When f is only one impulse. Then convolution is just translation"



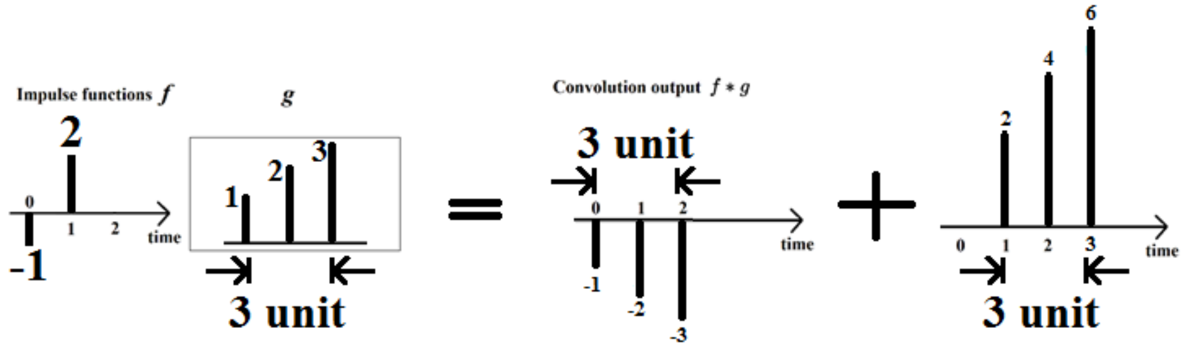
"Amplitude is just magnification"

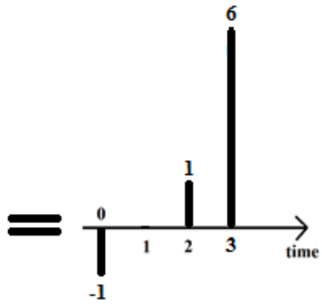


"Multiple impulses? Just add them together!"



"Don't forget the shape of g matters!"





EXAMPLE 8. $f = [1\ \ 2\ \ 3\ \ 4]$, $g = [1\ -1\ \ \ 1\,]$. Find $f * g$.
Answer
Consider $f * g$

Index	0	1	2	3	4	5
$f * g[1]$	1	2	3	4		
$f * g[2]$		-1	-2	-3	-4	
$f * g[3]$			1	2	3	4

Width of g

Length of f

Result: $f * g = f * g[1] + f * g[2] + f * g[3] = [1, 1,2,3, -1,4]$
Consider $g * f$

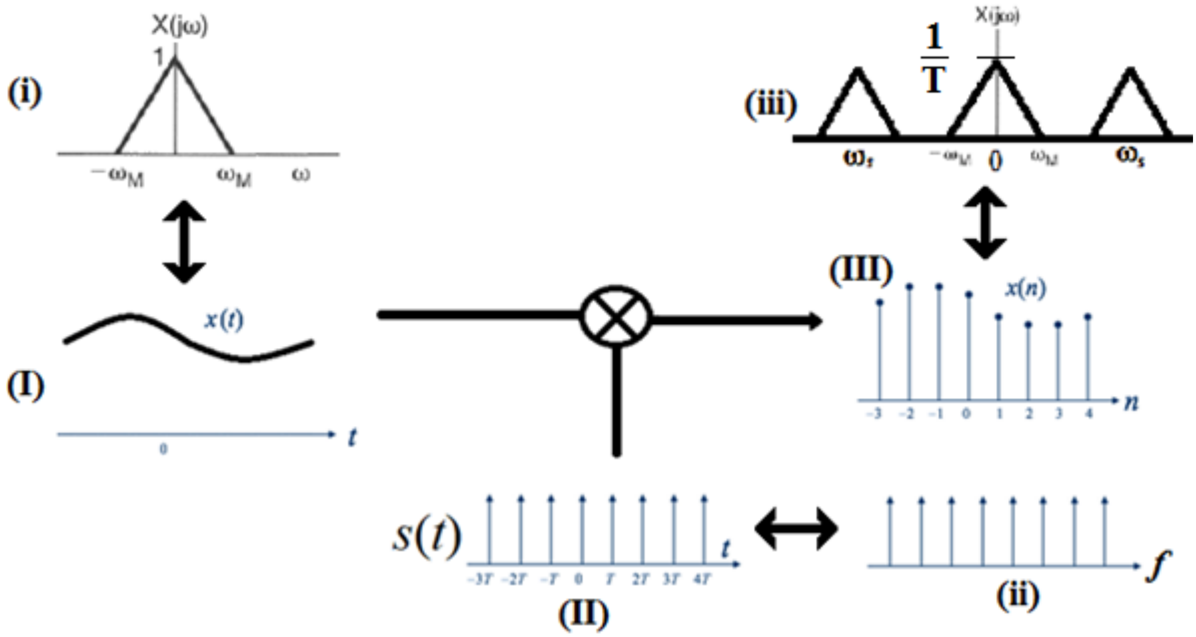
Index	0	1	2	3	4	5
$g * f[1]$	1	-1	1			
$g * f[2]$		2	-2	2		
$g * f[3]$			3	-3	3	
$g * f[4]$				4	-4	4

Width of f

Length of g

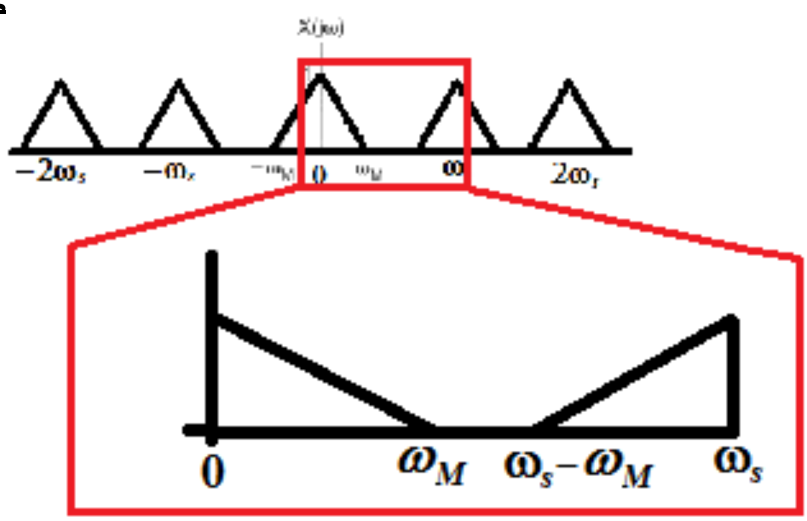
Also give you $[1, 1,2,3, -1,4]$, since $f * g = g * f$

Hence, the overall picture of the sampling process is shown below



(I) and (II) produce (III) through multiplication.
(i) and (ii) produce (iii) through convolution
Relationship between (I)-(i), (II)-(ii) and (III)-(iii) are time/frequency transform (Fourier series or Fourier Transform)

2b. Aliasing



Aliasing refers to the situation when two different signal are *indistinguishable* from each other. When the two triangle "not touching each other", there is no aliasing. To make sure there is no aliasing:

$$\omega_s - \omega_M \geq \omega_M$$

The boundary case is

$$\omega_s \geq 2\omega_M$$

In words: The sampling frequency has to be *at least* larger than the 2 times of the highest frequency of the signal.

Sampling Theorem (Shannon-Nyquist Sampling Theorem)

If $\omega_s \geq 2\omega_M$, we can collect all information about $x(t)$, and later use $x[n]$ to perfectly reconstruct $x(t)$. No information loss.
If $\omega_s < 2\omega_M$, we cannot recover $x(t)$ from $x[n]$ without any information loss. (We do not collect all information about $x(t)$ indeed).

EXAMPLE 9. Given $x(t) = \cos 2\pi 8t$, the sampling frequency is 4Hz, find $x[n]$.

$$x(nT) = \cos 2\pi 8nT$$
$$= \cos 2\pi 8n\frac{1}{4}$$
$$= \cos 2\pi 2n$$

n	…	-1	0	1	2	3	…
$x[n]$	1	1	1	1	1	1	1

$x[n] = \{\ldots, 1,1,1, \ldots\}$.

Comment: if you show somebody $x[n] = \{\ldots, 1,1,1, \ldots\}$, and ask what is the original $x(t)$. He will think that $x(t) = 1(t) = u(t) + u(-t)$. But the truth is $x(t) = \cos 2\pi 8t$!

EXAMPLE 10. Given $x(t) = \cos 2\pi 8t$, the sampling frequency is 16Hz, find $x[n]$.

$$x(nT) = \cos 2\pi 8nT$$
$$= \cos 2\pi 8n\frac{1}{16}$$
$$= \cos n\pi$$

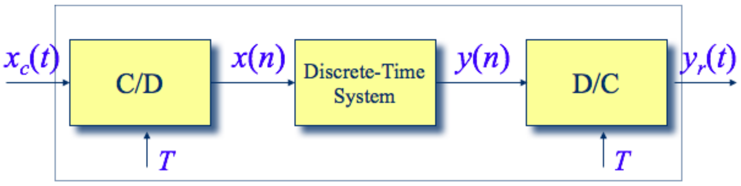
n	…	-1	0	1	2	3	…
$x[n]$	…	-1	1	-1	1	-1	…

$x[n] = \{\ldots, -1, 1, -1, \ldots\}$.

Comment: if you show somebody $x[n] = \{\ldots, -1, 1, -1, \ldots\}$, and ask what is the original $x(t)$. He will think that $x(t) = 1(t) = u(t) + u(-t)$. But the truth is $x(t) = \cos 2\pi 8t$!

3. Filtering System

The following describes the components of a digital signal processing (DSP) system.
DSP in block diagram:



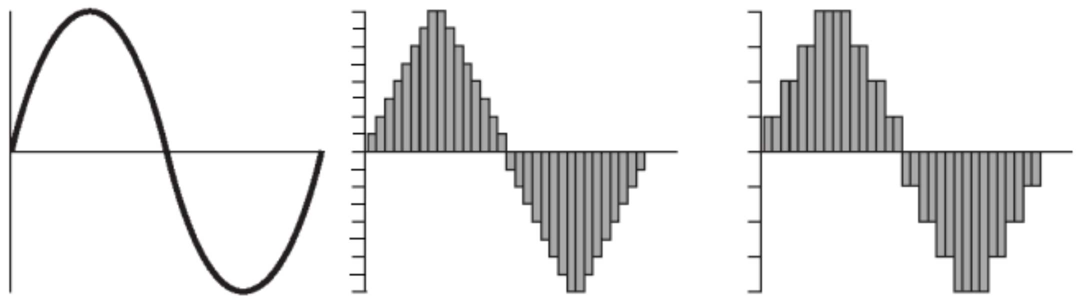
$x_c(t)$: continuous time signal
C/D: Continuous to Digital converter, (consists of sampler and encoder)
$x_c(t)$: continuous time signal
Discrete-Time System: a system in the form of $y[n] = \mathrm{G}\{x[n]\}$
D/C: Digital to continuous converter
T: sampling frequency

* In practice, there will be an anti-aliasing filter before the C/D block to make sure the sampling frequency is good enough for the purpose.

EXAMPLE 11. Distinguish the following: Continuous signal, discrete time signal, analog signal, digital signal.
Answer:

	Continuous Time	Discrete Time
Continuous amplitude	"Analog signal" "continuous-valued function of a continuous variable"	"Discrete (Time) signal" "sampled signal" "continuous-valued function of a discrete variable"
Discrete amplitude	"discrete-valued function of a continuous variable"	"Digital signal" "discrete-valued function of a discrete variable"

A C/D block consists of *sampler* and *encoder*. Sampling works on the "time-axis" while encoding works on the "amplitude-axis". "High resolution" means the grid size along the x-axis and/or y-axis is finer.



Analog **Digitized** **Digitized (lower precision)**

EXAMPLE 12. **(Inverse Fourier Transform)** Peter ask for a filter, that all the components with frequency higher than b Hz are removed.

(i) Find the frequency domain expression of the filer.
(ii) Find the time domain expression of the filter.
(iii) Comment the time domain expression of the filter.

Answer: Let $H(\omega)$ be the filter expression in the frequency domain.

$$H(\omega) = \begin{cases} 1 & \text{if } |\omega| \le 2\pi b \\ 0 & else \end{cases}$$

Let $h(n)$ be the filter expression in discrete time domain. To obtain $h(n)$ form $H(\omega)$, we need to use *Inverse Fourier Transform*.

- Why Fourier Transform: The spectrum $H(\omega)$ is clearly not a discrete bar plot but a continuous curve (i.e. the time domain signal is aperiodic!), we cannot use Fourier series.
- Why Inverse: We are now converting frequency back to time, not time to frequency!

Inverse Fourier Transform is

$$h(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} H(\omega) e^{j\omega n} d\omega$$

where ω_c is the *cut off frequency*. In this example, $\omega_c = 2\pi b$.

Hence $h(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{1}{2\pi}\frac{1}{jn} e^{j\omega n}\Big|_{-\omega_c}^{\omega_c} = \frac{1}{2\pi}\frac{1}{jn}\left(e^{j\omega_c n} - e^{-j\omega_c n}\right) = \frac{1}{\pi}\frac{1}{n}\sin\omega_c n$ and finally

$$h(n) = \frac{\omega_c}{\pi}\operatorname{sinc}\omega_c n$$

Comment: This filter has terms for time n < 0. Therefore it is a *non-causal filter*. The length of the filter is infinite. In practice, it is impossible to implement such infinite long filter. Hence such *ideal filter* is *not practical*.

EXAMPLE 13. Let say Peter want an ideal filter with cut off frequency of 10Hz. What is the filter expression in time and frequency domain?

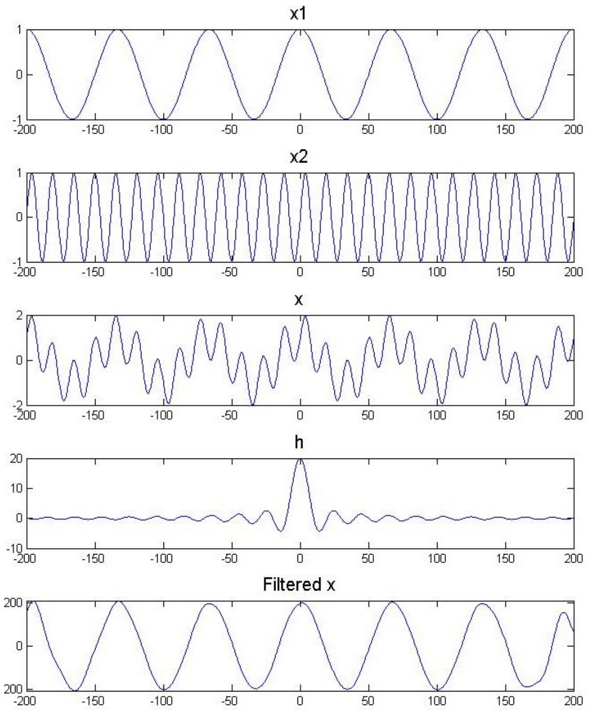
$$H(\omega) = \begin{cases} 1 & \text{if } |\omega| \le 20\pi \\ 0 & else \end{cases} \quad , \quad h[n] = 20\operatorname{sinc} 20\pi n$$

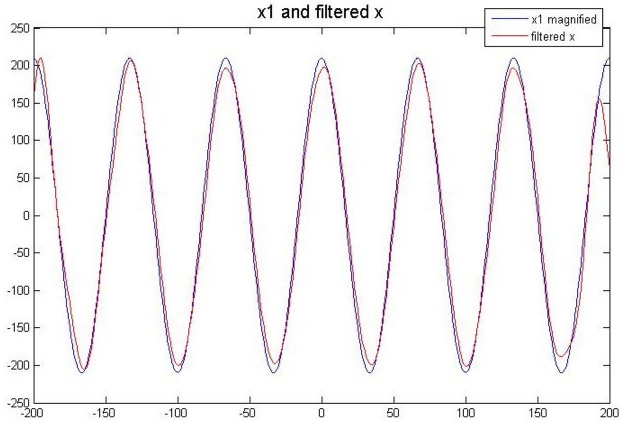
That is, for an input signal $x[n]$. The output $y[n]$ obtained by convoluting the input with the impulse respond $h[n]$ will have the effect of chopping of the signal with frequency higher than 10Hz.

EXAMPLE 14. (Computer Example) Perform filtering.
Consider we have two signals $x_1 = \cos 2\pi 3t$, $x_2 = \sin 2\pi 13t$. The two signal combined together to form the input signal $x = x_1 + x_2$. Now what we want to do is to "remove" the higher frequency component. Meaning that we consider the higher frequency component as noise. Consider we use a filter with cut off frequency at 10Hz, so $h[n] = 20\operatorname{sinc} 20\pi n$. Let *y* denoted the filtered *x*. Finally, assumes all wave are sampled using 200Hz, so $T = \frac{1}{200}$, and consider *n* = -200 to 200.

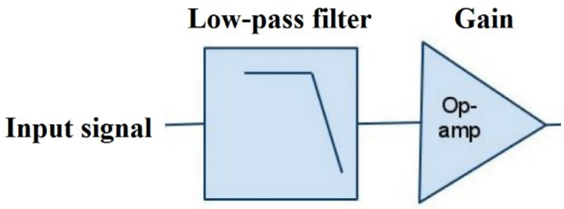
MATLAB Code

```
clcall
fs = 200;
T = 1/fs;
n = -200:200;
t = n*T;
f1 = 3;
f2 = 13;
x1 = cos(2*pi*f1*t);
x2 = sin(2*pi*f2*t);
x = x1+x2;
fc = 10;
h =20*sin(2*pi*fc*t)./(2*pi*fc*t);
h(find(isnan(h)))= 20;  % fill in the value of h when t = 0;
y = conv(x,h);
%plot the result
subplot(511),plot(n,x1),title('x1','fontsize',15)
subplot(512),plot(n,x2),title('x2','fontsize',15)
subplot(513),plot(n,x),title('x','fontsize',15)
subplot(514),plot(n,h),title('h','fontsize',15)
L_y = length(y); % for alignment
L_n = length(n); % for alignment
y_plot = y((L_y-L_n)/2: (L_y-L_n)/2+L_n-1) ;
subplot(515),plot(n,y_plot),title('Filtered x','fontsize',15),axis tight
%plot both x1 and filtered x
amp_y = max(y_plot);
x1_magnified = x1*amp_y;
figure,plot(n,x1_magnified,'b'),hold on, plot(n,y_plot,'r')
title('x1 and filtered x','fontsize',15), legend('x1 magnified','filtered x')
```





Comment: the filtered signal successfully removed the higher frequency component in x. And the result is highly similar to the original wave $x_1 = \cos 2\pi 3t$, but magnified about 200 times larger. So usually after filtering, an op-amp is used to magnify / diminish the signal amplitude.



EXAMPLE 15. (Optional, extra)(High pass filter) Consider high pass filtering with the following specific filter:

$$H(\omega) = \begin{cases} 0 & |\omega| \le \omega_c \\ e^{-\frac{j\omega M}{2}} & |\omega| \ge \omega_c \end{cases}$$

Answer.

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \left[\int_{\omega_c}^{\pi} e^{-\frac{j\omega M}{2}} e^{j\omega n} d\omega + \int_{-\pi}^{-\omega_c} e^{-\frac{j\omega M}{2}} e^{j\omega n} d\omega \right]$$

After many steps

$$h(n) = \operatorname{sinc}\left\{\pi\left(\frac{M}{2} - n\right)\right\} - \frac{\omega_c}{\pi} \operatorname{sinc}\left\{\omega_c\left(\frac{M}{2} - n\right)\right\}$$

EXAMPLE. 16 (Optional, extra). Perform the D/C conversion
For a continuous time signal $\boldsymbol{x_c(t)}$ with spectrum $\boldsymbol{X_c(\omega)}$. Sampling process creates signal $\boldsymbol{x}[n]$ with spectrum $\boldsymbol{X(\omega)}$. Now assume we have $\boldsymbol{x[n]}$ and $\boldsymbol{X(\omega)}$. We want to get back $\boldsymbol{x_c(t)}$.

Note that the relationship between $X_c(\omega)$ and $X(\omega)$ is: (see section 2!)

1. Along x-axis, $X(\omega)$ has multiple $X_c(\omega)$ located in $0\omega_s, \pm 1\omega_s, \pm 2\omega_s, \ldots$
2. Along y-axis, $X(\omega) = \frac{1}{T} X_c(\omega)$

- If the spectrum $X(\omega)$ is discrete, then $x_c(t)$ is just the Fourier series with coefficient just equal to the spectrum.
- If the spectrum $X(\omega)$ is continuous, then $x_c(t)$ is obtained by Inverse Fourier Transform.

$$
\begin{aligned}
x_c(t) &= \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} X_c(\omega) e^{j\omega t} d\omega \\
&= \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} T X(\omega) e^{j\omega t} d\omega \\
&= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \left(\sum_{n=-\infty}^{\infty} x_c(nT) e^{-j\omega nT} \right) e^{j\omega t} d\omega \\
&= \sum_{n=-\infty}^{\infty} x_c(nT) \left[\frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} e^{-j\omega nT} e^{j\omega t} d\omega \right]
\end{aligned}
$$

After some calculation, it can be shown that the square-bracket equal to sinc function and thus

$$
x_c(t) = \sum_{n=-\infty}^{\infty} x_c(nT) \operatorname{sinc} \frac{\pi}{T} (t - nT)
$$

Meaning: $x_c(t)$ can be reconstructed by interpolation (a method of constructing new data points within the range of a discrete set of known data points) using sinc basis.

It is important to know that, the reconstruction equation contains the term T (sampling period). So what the sampling theorem said is that:

> The sampling process turns a continuous time signal $x_c(t)$ to $x[n]$ with sampling frequency f_S, if the sampling frequency is at least 2 time higher than the bandwidth (maximum frequency content) of the signal, then there is no information lost. The $x_c(t)$ can be perfectly reconstructed from $x[n]$ using the sinc function.

-END OF Tutorial 3-

MEDE2500 Tutorial 4
2016-Nov-21
Z-transform and applications

What is Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Notice that if let $z = e^{j\omega}$

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

- when Δn is very small, it becomes Fourier Transform!
- $X\left(e^{j\omega}\right)$ or $X(\omega)$, are very useful.

<u>Z-transform</u>

Why Z-transform (Simple answer)

- Z-transform is a tool for analyzing discrete time signal and discrete time process

Why Z-transform (Mathematical answer)

- Fourier Transform does not converge on all sequence
- Z-transform links the theory of complex variable to discrete-time signal and system

Geometric series is very important for Z-transform

EXAMPLE 1. Derive the formula of the sums of geometric series

Answer: Consider the geometric series $1, r, r^2, \ldots, r^n, \ldots$. And let the sums $1 + r + \cdots r^n$ denoted as $S(n)$. Consider $rS(n) - S(n)$

$$\begin{aligned} rS(n) - S(n) &= r(1 + r + \cdots r^n) - (1 + r + \cdots r^n) \\ (r-1)S(n) &= (r + \cdots r^n + r^{n+1}) - (1 + r + \cdots r^n) \end{aligned}$$

$$(r-1)S(n) = r^{n+1} - 1$$

$$S(n) = \frac{r^{n+1} - 1}{r - 1}$$

$$S(n) = \frac{1 - r^{n+1}}{1 - r}$$

Assumes $|r| < 1$, when $n \to \infty$, $r^{n+1} \to 0$, thus

$$S(n) = \frac{1}{1 - r}$$

EXAMPLE 2. Find the Z-transform of the following

(i) $x[n] = [0,1,2,3]$

(ii) $[n] = \delta[n]$

(iii) $x[n] = u[n]$

(iv) $x[n] = a^n u[n]$

(v) $x[n] = -a^n u[-n-1]$

(vi) Compare (iv) and (v)

Answer

(i) $x[n] = [0,1,2,3]$, $X(z) = z^{-1} + 2z^{-2}3z^{-3}$

(ii) $x[n] = \delta[n]$, $X(z) = 1$

(iii) $x[n] = u[n]$

$$X(z) = \sum_{n=0}^{\infty} z^{-n}$$

Apply example 1 with $r = z^{-1}$

$$X(z) = \frac{1}{1 - z^{-1}}$$

(iv) $x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n$$

Apply example 1 with $r = az^{-1}$

$$X(z) = \frac{1}{1 - az^{-1}}$$

(v) $x[n] = -a^n u[-n-1]$

Based on $u[-n-1]$, the range change to $[-\infty \quad -1]$

$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

Tricks: let $m = -n$ to flip the range from $[-\infty \quad -1]$ to $[1 \quad \infty]$

$$X(z) = -\sum_{m=1}^{\infty} a^{-m} z^m$$

Tricks:

$$-\sum_{m=1}^{\infty} a^{-m}z^{m} = -\sum_{m=1}^{\infty} a^{-m}z^{m} - a^{0}z^{0} + a^{0}z^{0}$$

Thus

$$X(z) = 1 - \sum_{m=0}^{\infty} a^{-m}z^{m}$$

Apply example 1 with $r = (a^{-1}z)$

$$X(z) = 1 - \frac{1}{1 - a^{-1}z}$$

(vi) Compare (iii) and (iv)

$$x_1[n] = a^n u[n] \qquad X_1(z) = \frac{1}{1 - az^{-1}}$$

$$x_2[n] = -a^n u[-n-1] \qquad X_2(z) = 1 - \frac{1}{1 - a^{-1}z}$$

Note that

$$X_1(z) = X_1(z)\frac{z}{z} = \frac{1}{1 - az^{-1}}\frac{z}{z} = \frac{z}{z-a}$$

$$\begin{aligned} X_2(z) &= 1 - \frac{1}{1 - a^{-1}z} \\ &= 1 - \frac{a}{a - z} \\ &= 1 - \frac{-a}{z - a} \\ &= \frac{z-a}{z-a} - \frac{-a}{z-a} \\ &= \frac{z}{z-a} \end{aligned}$$

That is, $\mathcal{Z}\{a^n u[n]\} = \mathcal{Z}\{-a^n u[-n-1]\}$!!

To avoid confusion, we should always mention the condition of z called ROC (Region of convergence).

For $x_1[n]$, $|az^{-1}| < 1$ and thus ROC is $|z| > |a|$. For $x_2[n]$, $|a^{-1}z| < 1$ and thus ROC is $|z| < |a|$.

Hence even $X_1(z) = X_2(z)$ after some manipulation, their ROC is different.

EXAMPLE 3. Find the Z-transform $x[n] = \cos[kn]u[n]$

$$X(z) = \sum_{n=0}^{\infty} \cos[kn]\, z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} \frac{\exp[jkn] + \exp\,[-jkn]}{2} z^{-n}$$

$$X(z) = \frac{1}{2}\left[\sum_{n=0}^{\infty} \left(\mathrm{e}^{jk} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(\mathrm{e}^{-jk} z^{-1}\right)^n\right]$$

$$X(z) = \frac{1}{2}\left[\frac{1}{1 - e^{jk} z^{-1}} + \frac{1}{1 - e^{-jk} z^{-1}}\right]$$

$$X(z) = \frac{1}{2} \frac{1 - e^{-jk} z^{-1} + 1 - e^{jk} z^{-1}}{(1 - e^{jk} z^{-1})(1 - e^{-jk} z^{-1})}$$

$$X(z) = \frac{1}{2} \frac{2 - (e^{-jk} + e^{jk}) z^{-1}}{1 - (e^{-jk} + e^{jk}) z^{-1} + z^{-2}}$$

$$X(z) = \frac{1}{2} \frac{2 - 2\cos k\, z^{-1}}{1 - (e^{-jk} + e^{jk}) z^{-1} + z^{-2}}$$

$$X(z) = \frac{1 - \cos k\, z^{-1}}{1 - 2\cos k\, z^{-1} + z^{-2}}$$

EXAMPLE 4. Find the inverse Z-transform of (i) $X(z) = \frac{1}{1-az^{-2}}$ (ii) $X(z) = \frac{1}{az^{-1}+bz^{-2}}$

Answer

(i)

$$X(z) = \frac{1}{1 - az^{-2}}$$

$$= \frac{1}{(1 + \sqrt{a}z^{-1})(1 - \sqrt{a}z^{-1})}$$

Perform partial fraction

$$= \frac{\frac{1}{2}}{1 + \sqrt{a}z^{-1}} + \frac{\frac{1}{2}}{1 - \sqrt{a}z^{-1}}$$

Since

$$x_1[n] = a^n u[n] \overset{z}{\leftrightarrow} X(z) = \frac{1}{1 - az^{-1}}$$

Therefore

$$x(n) = \frac{1}{2}\left(\left(-\sqrt{a}\right)^n u[n] + \sqrt{a}^n u[n]\right)$$

(ii)

$$X(z) = \frac{1}{az^{-1} + bz^{-2}}$$

$$X(z) = \frac{1}{a}\frac{1}{z^{-1}}\frac{1}{1 + \frac{b}{a}z^{-1}}$$

After partial fraction

$$X(z) = \frac{1}{a}\left[\frac{\frac{1}{1}}{z^{-1}} + \frac{\frac{1}{-a/b}}{1 + \frac{b}{a}z^{-1}}\right]$$

$$X(z) = \frac{1}{a}\left[z - \frac{b}{a}\frac{1}{1 + \frac{b}{a}z^{-1}}\right]$$

Then use table look up to find the inverse Z-transform.

EXAMPLE 6. (Fast Partial Fraction – Heaviside Cover-up Method) Find the the unknowns A, B and C in the following equality.

$$f(x) = \frac{1}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

Answer. Multiply the whole expression by $f(x)$

$$1 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

Put $x = -1$ to solve for A

$$A = \frac{1}{(-1+2)(-1+3)} = \frac{1}{2}$$

Similarly

$$B = \frac{1}{(-2+1)(-2+3)} = \frac{-1}{1}$$

$$C = \frac{1}{(-3+1)(-3+2)} = \frac{1}{2}$$

Therefore

$$f(x)=\frac{\frac{1}{2}}{x+1}+\frac{-1}{x+2}+\frac{\frac{1}{2}}{x+3}$$

Note that

$$A=f(x)(x+1)|_{x=-1}=\frac{1}{2}$$

$$B=f(x)(x+2)|_{x=-2}=-\frac{1}{1}$$

$$C=f(x)(x+3)|_{x=-3}=\frac{1}{2}$$

EXAMPLE 7. Perform the fast partial fraction on the following

$$f(x)=\frac{1}{(x+a)(x+b)(x+c)}$$

Answer

$$f(x)=\frac{\frac{1}{(b-a)(c-a)}}{x+a}+\frac{\frac{1}{(a-b)(c-b)}}{x+b}+\frac{\frac{1}{(a-c)(b-c)}}{x+c}$$

EXAMPLE 8. Find the partial fraction of

$$f(x)=\frac{x}{x^2-1}$$

Answer:

$$\begin{aligned}f(x)&=\frac{x}{(x+1)(x-1)}\\&=\frac{\frac{-1}{-1-1}}{x+1}+\frac{\frac{1}{1+1}}{x-1}\\&=\frac{\frac{1}{2}}{x+1}+\frac{\frac{1}{2}}{x-1}\end{aligned}$$

EXAMPLE 9. Find the inverse Z-transform of

$$f(z^{-1}) = \frac{z^{-1} - 1}{z^{-2} - z^{-1} - 6}$$

Answer:

$$\begin{aligned} f(z^{-1}) &= \frac{z^{-1} - 1}{(z^{-1} - 3)(z^{-1} + 2)} \\ &= \frac{\frac{3-1}{3+2}}{z^{-1} - 3} + \frac{\frac{-2-1}{-2-3}}{z^{-1} + 2} \\ &= \frac{\frac{2}{5}}{z^{-1} - 3} + \frac{\frac{3}{5}}{z^{-1} + 2} \end{aligned}$$

Change to standard form

$$f(z^{-1}) = \frac{2}{-15}\frac{1}{1 - \frac{1}{3}z^{-1}} + \frac{3}{10}\frac{1}{1 - (-1)z^{-1}}$$

The inverse Z-transform is thus

$$\mathcal{Z}^{-1}[f(z^{-1})] = \frac{2}{-15}\left(\frac{1}{3}\right)^{n} u[n] + \frac{3}{10}(-1)^{n} u[n]$$

EXAMPLE 10. (Shift theorem of Z-transform) Find the Z-transform of $x(n-1)$

$$\sum_{n=0}^{\infty} x(n-1)z^{-n}$$

Let $n - 1 = m$

$$= \sum_{m+1=0}^{\infty} x(m)z^{-(m+1)}$$

$$= \sum_{m=-1}^{\infty} x(m)z^{-m}z^{-1}$$

$$= z^{-1} \sum_{m=-1}^{\infty} x(m)z^{-m}$$

Assumes the signal is causal $x(n) = 0$ for $n < 0$.

$$= z^{-1} \sum_{m=0}^{\infty} x(m)z^{-m}$$

Since $X(z) = \sum_{m=0}^{\infty} x(m)z^{-m}$, hence

$$\mathcal{Z}[x(n-1)] = z^{-1}X(z)$$

In general

$$\mathcal{Z}[x(n-k)] = z^{-k}X(z)$$

Application of Z-transform.

Consider input signal $x[n]$ and system $h[n]$, the output is the convolution $y[n] = x[n] * h[n]$.

In z-domain, the relationship becomes a simple multiplication $Y(z) = X(z)H(z)$.

EXAMPLE 11. Consider the system $h(n) = [0,1,-2,3]$ with input signal $x(n) = [1,2,3]$. Find the output $y[n]$.

Method 1. By convolution $y[n] = x[n] * h[n]$

n	0	1	2	3	4	5
$h(0)x(n)$	0	0	0			
$h(1)x(n)$		1	2	3		
$h(2)x(n)$			-2	-4	-6	
$h(3)x(n)$				3	6	9
Total	0	1	0	2	0	9

Hence $y[n] = [0,1,0,2,0,9]$

Method 2. By Z-transform

$$H(z) = z^{-1} - 2z^{-2} + 3z^{-3} \quad X(z) = 1 + 2z^{-1} + 3z^{-2}$$

Since $Y(z) = X(z)H(z)$, thus

$$Y(z) = (z^{-1} - 2z^{-2} + 3z^{-3})(1 + 2z^{-1} + 3z^{-2})$$

$$Y(z) = 0z^{-0} + 1z^{-1} + 0z^{-2} + 2z^{-3} + 0z^{-4} + 9z^{-5}$$

So $y(n) = [0,1,0,2,0,9]$

EXAMPLE 12. Consider the system $h(n) = \left[\frac{1}{2}, \frac{1}{2}\right]$with input signal $x(n) = \cos(2\pi \frac{10}{7} n\ )$u(n).

Find the output $y[n]$.

Method 1. Convolution. $y[n] = x[n] * h[n]$

n	0	1	2	3	4	…
$h(0)x(n)$	$\frac{1}{2}\cos(0)$	$\frac{1}{2}\cos(2\pi\frac{10}{7})$	$\frac{1}{2}\cos(2\pi\frac{20}{7})$	$\frac{1}{2}\cos(2\pi\frac{30}{7})$	$\frac{1}{2}\cos(2\pi\frac{40}{7})$	…
$h(1)x(n)$		$\frac{1}{2}\cos(0)$	$\frac{1}{2}\cos(2\pi\frac{10}{7})$	$\frac{1}{2}\cos(2\pi\frac{20}{7})$	$\frac{1}{2}\cos(2\pi\frac{30}{7})$	…
Total	$\frac{1}{2}\cos(0)$	$\frac{x(1)+x(0)}{2}$	$\frac{x(2)+x(1)}{2}$	$\frac{x(3)+x(2)}{2}$	$\frac{x(4)+x(3)}{2}$	…

Conclusion: not very useful.

Method 2. Use *frequency response*

$$H(z) = \frac{1}{2} + \frac{1}{2}z^{-1}$$

The formula for LTI system with sinusoidal inputs

$$y(n) = \left|H\left(e^{j\omega}\right)_{\omega=\omega_0}\right| \cos\left(\omega_0 n + \angle H\left(e^{j\omega}\right)_{\omega=\omega_0}\right)$$

For $\cos\left(2\pi\frac{10}{7}n\right) = \cos w_0 n$, $\omega_0 = 2\pi\frac{10}{7}$

$$H\left(e^{j\omega}\right) = \frac{1}{2} + \frac{1}{2}e^{-j\omega}$$
$$= \frac{1}{2}\left(1 + e^{-j\omega}\right)$$

Tricks

$$= \frac{1}{2}\left(1 + e^{-j\omega}\right)\frac{e^{j\omega/2}}{e^{j\omega/2}}$$
$$= \frac{1}{2}\left(e^{j\omega/2} + e^{-j\omega/2}\right)\frac{1}{e^{j\omega/2}}$$

Apply Euler's Formula

$$H\left(e^{j\omega}\right) = \cos\left(\frac{\omega}{2}\right)\frac{1}{e^{j\omega/2}}$$

Or

$$H\left(e^{j\omega}\right) = \cos\left(\frac{\omega}{2}\right)e^{-j\omega/2}$$

Note that $H\left(e^{j\omega}\right)$ is a complex number, so it has magnitude and phase.

To obtain the magnitude, take the absolute value of $H(e^{j\omega})$. Since $|e^{j\theta}| = 1$ for any θ,

$$|H(e^{j\omega})| = \left|\cos\left(\frac{\omega}{2}\right)\right|$$

Put $\omega = \omega_0 = 2\pi\frac{10}{7}$

$$|H(e^{j\omega_0})| = \left|\cos\left(\frac{10}{7}\pi\right)\right|$$

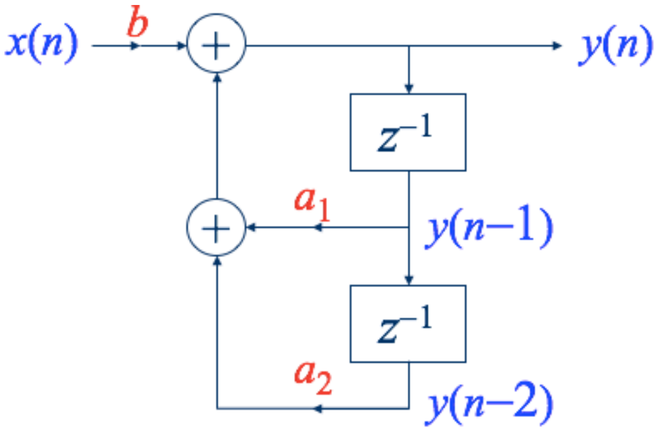
To obtain the phase angle,

$$\angle H(e^{j\omega})_{\omega=\omega_0} = \angle e^{-j\omega_0/2} = \frac{\omega_0}{2} = \frac{10}{7}\pi$$

Hence the output is

$$y(n) = \left|\cos\left(\frac{10}{7}\pi\right)\right| \cos\left(\omega_0 n + \frac{10}{7}\pi\right)$$

EXAMPLE 13. Consider the following block diagram, assume $a_1 = -1$, $a_2 = 1$, $b = 2$



(a) Write down the system function in time domain.

(b) Find the Z-transform and hence the system transfer function.

(c) Find the output for input $x[n] = 2[u(n) - u(n-2)] - \sin\left(2\pi\frac{6}{f_s}n + \frac{\pi}{2}\right)$ for $f_s = 36$.

Answer.

(a) $y[n] = a_1 y[n-1] + a_2 y[n-2] + bx[n]$

(b) Perform Z-transform

$$Y(z) = a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + bX(z)$$

Rearrange

$$Y(z)(1 - a_1 z^{-1} - a_2 z^{-2}) = bX(z)$$

The transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - a_1 z^{-1} - a_2 z^{-2}}{b}$$

(c) For $x[n] = 2[u(n) - u(n-2)] - \sin\left(2\pi \frac{6}{f_s}\pi + \frac{\pi}{2}\right)$

Let $x_1[n] = 2[u(n) - u(n-2)]$ and $x_2[n] = \sin\left(2\pi \frac{6}{f_s} n + \frac{\pi}{2}\right)$

Output $y[n]$ consists of two parts, the part $y_1[n]$ form $x_1[n]$ and the part $y_2[n]$ from $x_2[n]$

$$X_1(z) = 2\left[\frac{1}{1 - z^{-1}} - \frac{z^{-2}}{1 - z^{-1}}\right]$$

$$= 2\frac{1 - z^{-2}}{1 - z^{-1}}$$

$$= 2(1 + z^{-1})$$

Hence

$$Y_1(z) = H(z)X(z) = \frac{1 - a_1 z^{-1} - a_2 z^{-2}}{b} 2(1 + z^{-1})$$

$$= \frac{2}{b}[1z^0 + (1 - a_1)z^{-1} + (-a_1 - a_2)z^{-2} + (-a_2)z^{-3}]$$

Put $a_1 = -1, a_2 = 1, b = 2$

$$Y_1(z) = 1 + 2z^{-1} - z^{-3}$$

Therefore

$$y_1[n] = [1 \quad 2 \quad -1]$$

$$H(z) = \frac{1 - a_1 z^{-1} - a_2 z^{-2}}{b}$$

Put $a_1 = -1, a_2 = 1, b = 2, z = e^{j\omega}$

$$H\left(e^{j\omega}\right) = \frac{1 + e^{-j\omega} - e^{-2j\omega}}{2}$$

Multiply with $e^{j\omega}/e^{j\omega}$

$$H\left(e^{j\omega}\right) = \frac{e^{j\omega} + 1 - e^{-j\omega}}{2e^{j\omega}}$$

Thus

$$H\left(e^{j\omega}\right) = \frac{1 + 2j\sin\omega}{2e^{j\omega}}$$

$$\left|H\left(e^{j\omega}\right)\right| = \frac{\sqrt{1 + 4\sin^2 \omega}}{2}$$

$$\left|H\left(e^{j\omega_0}\right)\right| = \frac{\sqrt{1 + 4\sin^2 \frac{\pi}{3}}}{2} = 1$$

The angle

$$H\left(e^{j\omega}\right) = \frac{1 + \cos\omega - j\sin\omega - \cos 2\omega + j\sin 2\omega}{2}$$
$$= \frac{1 + \cos\omega - \cos 2\omega + j\,(\sin 2\omega - \sin\omega)}{2}$$

$$\angle H\left(e^{j\omega}\right)_{\omega=\omega_0} = \tan^{-1}\frac{\sin 2\omega_0 - \sin\omega_0}{1 + \cos\omega_0 - \cos 2\omega_0}$$

Hence the output of the sinusoid is

$$\sin\left(2\pi\frac{6}{f_s}\pi + \frac{\pi}{2} + \tan^{-1}\frac{\sin 2\omega_0 - \sin\omega_0}{1 + \cos\omega_0 - \cos 2\omega_0}\right)$$

-END-