Discrete Total Variation

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Recall continuous Total Variation

- Ω is a closed bounded subset of $\mathbb{R}^2$.
- $u(x, y) : \Omega \to \mathbb{R}$.
- Total variation within the domain Ω

$$\|u\|_{\mathrm{BV}(\Omega)} := \int_{\Omega} \|\nabla u(x, y)\| \, dxdy = \int_{\Omega} \sqrt{\left(\frac{\partial u(x, y)}{\partial x}\right)^2 + \left(\frac{\partial u(x, y)}{\partial y}\right)^2} \, dxdy,$$

  where BV(Ω) denotes bounded variation within the domain Ω.
- The subgradient (Gateaux derivative) of $\|u\|_{\mathrm{BV}(\Omega)}$ is a nonlinear Laplacian

$$-\nabla \cdot \left(\frac{\nabla u}{\|\nabla u\|}\right).$$

Discretizing single-variable total variation

$$\|u\|_{\mathrm{BV}(\Omega)} := \int_\Omega \|\nabla u(x)\| \, dx = \int_\Omega \sqrt{\left(\frac{du(x)}{dx}\right)^2} dx. = \int_\Omega \left|\frac{du(x)}{dx}\right| dx.$$

- Finite discretization:
  - 1st-order forward difference $\frac{du(x)}{dx} \approx \frac{u(x+h)-u(x)}{h}$

$$\|u\|_{\mathrm{TV}} = \frac{1}{h} \sum_{i \in \Omega} \left|u_{i+1} - u_i\right|$$

  - 1st-order backward difference $\frac{du(x)}{dx} \approx \frac{u(x)-u(x-h)}{h}$

$$\|u\|_{\mathrm{TV}} = \frac{1}{h} \sum_{i \in \Omega} \left|u_i - u_{i-1}\right|$$

  - 2nd-order central difference $\frac{du(x)}{dx} \approx \frac{u(x+h)-2u(x)+u(x-h)}{h^2}$

$$\|u\|_{\mathrm{TV}} = \frac{1}{h^2} \sum_{i \in \Omega} \left|u_{i+1} - 2u_i + u_{i-1}\right|$$

Discretizing two-variable total variation by forward difference

$$\|u\|_{\mathrm{BV}(\Omega)} := \int_\Omega \|\nabla u(x,y)\|\, dxdy = \int_\Omega \sqrt{\Big(\frac{\partial u(x,y)}{\partial x}\Big)^2 + \Big(\frac{\partial u(x,y)}{\partial y}\Big)^2}\, dxdy,$$

- By forward difference

$$\begin{aligned}\sqrt{\Big(\frac{\partial u(x,y)}{\partial x}\Big)^2 + \Big(\frac{\partial u(x,y)}{\partial y}\Big)^2} &\approx \sqrt{(\frac{u_{i,j+1} - u_{i,j}}{h})^2 + (\frac{u_{i+1,j} - u_{i,j}}{h})^2} \\ &= \frac{1}{h}\sqrt{(u_{i,j+1} - u_{i,j})^2 + (u_{i+1,j} - u_{i,j})^2}\end{aligned}$$

- Discrete TV is then

$$\|u\|_{\mathrm{TV}} = \frac{1}{h} \sum_{(i,j)\in\Omega} \sqrt{(u_{i,j+1} - u_{i,j})^2 + (u_{i+1,j} - u_{i,j})^2}$$

Such TV is also called Isotropic TV in image processing.

Discrete gradient, isotropic TV and Anisotropic TV

- Two types of discrete forward difference 2D TV

$$\|u\|_{\mathrm{TV}} = \frac{1}{h} \sum_{(i,j)\in\Omega} \sqrt{(u_{i,j+1} - u_{i,j})^2 + (u_{i+1,j} - u_{i,j})^2} \quad \text{Isotropic TV}$$

$$\|u\|_{\mathrm{TV}} = \frac{1}{h} \sum_{(i,j)\in\Omega} |u_{i,j+1} - u_{i,j}| + |u_{i+1,j} - u_{i,j}|. \quad \text{Anisotropic TV}$$

- The term anisotropic is used in the sense of anisotropic diffusion in image processing.
- If we define the discrete approximation of gradient ∇u as

$$\text{continuous gradient} = \nabla u(x,y) \approx \begin{bmatrix} u_{i,j+1} - u_{i,j} \\ u_{i,i+1} - u_{i,j} \end{bmatrix} = \begin{bmatrix} D_y u \\ D_x u \end{bmatrix} = \text{discrete gradient},$$

Then isotropic TV is ℓ_2 norm of the discrete gradient, and anisotropic TV is ℓ_1 norm of the discrete gradient.

2nd-order central difference on 2D TV

- It is possible to formulate more complicated version of discrete 2D TV.
- Isotropic TV with 2nd-order central difference approximation

$$\begin{aligned}\|u\|_{\mathrm{TV}} &= \sum_{(i,j)\in\Omega} \sqrt{\left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}\right)^2 + \left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right)^2} \\ &= \frac{1}{h^2} \sum_{(i,j)\in\Omega} \sqrt{\Big(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\Big)^2 + \Big(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\Big)^2}\end{aligned}$$

- Anisotropic TV with 2nd-order central difference approximation

$$\|u\|_{\mathrm{TV}} = \frac{1}{h^2} \sum_{(i,j)\in\Omega} \Big|u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\Big| + \Big|u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\Big|$$

Avoid central difference approximation: it may not work

- It is possible for the central-difference to be 0 but the image is not having zero variation.
- For example, consider
$$u = [0.7, 0.5, 0.3].$$
- Forward difference TV
$$\|u\|_{\mathrm{TV}} = |0.5 - 0.7| + |0.3 - 0.5| = 0.4 \neq 0.$$
- Central difference approximation
$$\|u\|_{\mathrm{TV}} = |0.7 - 2 \times 0.5 + 0.3| = 0.$$
- In practise: avoid central-difference and stick with forward / backward difference when approximating TV.

Subdifferential of (forward difference) discrete TV

- Now consider the (forward difference) isotropic TV

$$\|u\|_{\mathrm{TV}} = \frac{1}{h} \sum_{(i,j)\in\Omega} \sqrt{(u_{i,j+1} - u_{i,j})^2 + (u_{i+1,j} - u_{i,j})^2}$$

  what is $\partial_u \|u\|_{\mathrm{TV}}$?

- Consider $\partial_{u_{p,q}} \|u\|_{\mathrm{TV}}$. Notice that there are 3 terms in the sum containing $u_{p,q}$

$$\begin{array}{lll} p = i & q = j & \sqrt{(u_{i,j+1} - u_{i,j})^2 + (u_{i+1,j} - u_{i,j})^2} \\ p = i & q = j - 1 & \sqrt{(u_{i,j} - u_{i,j-1})^2 + (u_{i+1,j-1} - u_{i,j-1})^2} \\ p = i - 1 & q = j & \sqrt{(u_{i-1,j+1} - u_{i-1,j})^2 + (u_{i,j} - u_{i-1,j})^2} \end{array}$$

  Therefore, $\partial_{u_{p,q}} \|u\|_{\mathrm{TV}}$ will be a sum of 3 subdifferential.

The subdifferential of the 3 terms

$$
\begin{aligned}
\partial\sqrt{(u_{i,j+1}-u_{i,j})^2+(u_{i+1,j}-u_{i,j})^2} &= \frac{-2(u_{i,j+1}-u_{i,j})-2(u_{i+1,j}-u_{i,j})}{\sqrt{(u_{i,j+1}-u_{i,j})^2+(u_{i+1,j}-u_{i,j})^2}} \\
\partial\sqrt{(u_{i,j}-u_{i,j-1})^2+(u_{i+1,j-1}-u_{i,j-1})^2} &= \frac{2(u_{i,j}-u_{i,j-1})}{\sqrt{(u_{i,j}-u_{i,j-1})^2+(u_{i+1,j-1}-u_{i,j-1})^2}} \\
\partial\sqrt{(u_{i-1,j+1}-u_{i-1,j})^2+(u_{i,j}-u_{i-1,j})^2} &= \frac{2(u_{i,j}-u_{i-1,j})}{\sqrt{(u_{i-1,j+1}-u_{i-1,j})^2+(u_{i,j}-u_{i-1,j})^2}}
\end{aligned}
$$

Assuming the denominator is nonzero.

Case of $\sqrt{0^2}$

$$
\begin{array}{llll}
p=i & q=j & \sqrt{(u_{i,j+1}-u_{i,j})^2+(u_{i+1,j}-u_{i,j})^2} & =\sqrt{2}|u_{i,j}-c| \quad c=u_{i,j+1}=u_{i+1,j} \\
p=i & q=j-1 & \sqrt{(u_{i,j}-u_{i,j-1})^2+(u_{i+1,j-1}-u_{i,j-1})^2} & =|u_{i,j}-c| \quad c=u_{i,j-1} \\
p=i-1 & q=j & \sqrt{(u_{i-1,j+1}-u_{i-1,j})^2+(u_{i,j}-u_{i-1,j})^2} & =|u_{i,j}-c| \quad c=u_{i-1,j}
\end{array}
$$

- Illustration: for $\sqrt{(u_{i,j+1}-u_{i,j})^2+(u_{i+1,j}-u_{i,j})^2}=\sqrt{0^2}$, it means both bracket $=0$, which implies $u_{i,j+1}=u_{i+1,j}=: c$. Thus

$$
\begin{aligned}
\sqrt{(u_{i,j+1}-u_{i,j})^2+(u_{i+1,j}-u_{i,j})^2} &= \sqrt{2(c-u_{i,j})^2} \\
&= \sqrt{2}\sqrt{(u_{i,j}-c)^2} \\
&= \sqrt{2}|u_{i,j}-c|.
\end{aligned}
$$

- In this case the subdifferential of is in the form of

$$
\partial_x|x-c| = \operatorname{sign}(x-c) = \begin{cases} 1 & x>c \\ -1 & x<c \\ [-1,1] & x=c \end{cases}
$$

The subdifferential of discrete (forward difference) 2D TV

- Case $\sqrt{x^2}, x \neq 0$

$$
\begin{aligned}
\partial\sqrt{(u_{i,j+1}-u_{i,j})^2+(u_{i+1,j}-u_{i,j})^2} &= \frac{-2(u_{i,j+1}-u_{i,j})-2(u_{i+1,j}-u_{i,j})}{\sqrt{(u_{i,j+1}-u_{i,j})^2+(u_{i+1,j}-u_{i,j})^2}}\\
\partial\sqrt{(u_{i,j}-u_{i,j-1})^2+(u_{i+1,j-1}-u_{i,j-1})^2} &= \frac{2(u_{i,j}-u_{i,j-1})}{\sqrt{(u_{i,j}-u_{i,j-1})^2+(u_{i+1,j-1}-u_{i,j-1})^2}}\\
\partial\sqrt{(u_{i-1,j+1}-u_{i-1,j})^2+(u_{i,j}-u_{i-1,j})^2} &= \frac{2(u_{i,j}-u_{i-1,j})}{\sqrt{(u_{i-1,j+1}-u_{i-1,j})^2+(u_{i,j}-u_{i-1,j})^2}}
\end{aligned}
$$

- Case $\sqrt{x^2}, x = 0$

$$
\begin{aligned}
\partial\sqrt{(u_{i,j+1}-u_{i,j})^2+(u_{i+1,j}-u_{i,j})^2} &= \sqrt{2}\mathrm{sign}(u_{i,j}-c), c = u_{i,j+1} = u_{i+1,j}\\
\partial\sqrt{(u_{i,j}-u_{i,j-1})^2+(u_{i+1,j-1}-u_{i,j-1})^2} &= \mathrm{sign}(u_{i,j}-u_{i,j-1})\\
\partial\sqrt{(u_{i-1,j+1}-u_{i-1,j})^2+(u_{i,j}-u_{i-1,j})^2} &= \mathrm{sign}(u_{i,j}-u_{i-1,j})
\end{aligned}
$$

- The subdifferential of $\|u\|_{\mathrm{TV}}$ is to sum all the three terms over all (i, j).

ROF problem with discrete TV

- Continuous ROF problem (penalty form, constant ignored)

$$\min_u \int_\Omega \|\nabla u\| dxdy + \lambda \int_\Omega \frac{1}{2}(u-u_0)^2 dxdy \quad \text{s.t.} \quad \int_\Omega (u-u_0) dxdy = 0$$

- Discrete ROF problem (using isotropic discrete TV)

$$\min_u \frac{1}{h} \sum_{(i,j)\in\Omega} \sqrt{(u_{i,j+1}-u_{i,j})^2 + (u_{i+1,j}-u_{i,j})^2} + \frac{\lambda}{2}\|u-u_0\|_2^2$$

  note that we ignored the constraint in the continuous case.

- Discrete ROF problem in other notation

$$\min_u \frac{1}{h} \sum_{(i,j)\in\Omega} \left\| \begin{bmatrix} u_{i,j+1}-u_{i,j} \\ u_{i+1,j}-u_{i,j} \end{bmatrix} \right\|_2 + \frac{\lambda}{2}\|u-u_0\|_2^2$$

Smoothing

- The expression $\sqrt{x^2}$ is not smooth. Historically, people consider $\sqrt{x^2+\epsilon}$, where $\epsilon > 0$ is a small number to avoid the zero case.
- Similarly, we can consider the smoothed discrete isotropic 2D TV

$$\min_u \frac{1}{h} \sum_{(i,j)\in\Omega} \sqrt{(u_{i,j+1}-u_{i,j})^2 + (u_{i+1,j}-u_{i,j})^2 + \epsilon} + \frac{\lambda}{2}\|u-u_0\|_2^2$$

- The advantage of smoothing is that we do not need to deal with the singular case anymore.
- The disadvantage of smoothing is that we are now introducing a new parameter to tune.

Last page - summary

- 1st-order forward difference approximation of 2D TV

$$\|u\|_{\mathrm{TV}} = \frac{1}{h} \sum_{(i,j)\in\Omega} \sqrt{(u_{i,j+1} - u_{i,j})^2 + (u_{i+1,j} - u_{i,j})^2} \quad \text{Isotropic TV}$$

$$\|u\|_{\mathrm{TV}} = \frac{1}{h} \sum_{(i,j)\in\Omega} |u_{i,j+1} - u_{i,j}| + |u_{i+1,j} - u_{i,j}|. \qquad \text{Anisotropic TV}$$

- Caution in using 2nd-order central difference to approximate TV
- Subdifferential of isotropic TV
- Smoothing

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