Solving piece-wise linear equation

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Content

$$\sum_{i=1}^{n} c_i \big[a_i x + b_i\big]_+ = 1$$

Break point $\frac{-a_i}{b_i}$

Sorting-based algorithm

Piece-wise linear function

- A general form of piece-wise linear function

$$f(x) = \sum_{i=1}^{n} c_i \max\left\{0, a_i x + b_i\right\} - d,$$

  where a_i, b_i, c_i and d are given real numbers.

- Let $[\,\cdot\,]_+ \coloneqq \max\{0\,,\,\cdot\}$, then $f(x) = 0$ is the same as

$$\sum_{i=1}^{n} c_i \big[a_i x + b_i\big]_+ = d.$$

- This PDF: discuss how to solve for the root x.

Where does piece-wise linear function appear

- Discrete optimization
  Piece-wise linear function appears naturally in many discrete optimization problems.

- Projection onto polyhedral set. e.g., simplex, ℓ_1 ball
  For example, projecting a vector in $\mathbb{R}^n$ onto the convex set

$$\left\{ \boldsymbol{x} \mid \boldsymbol{x} \geq 0, \langle \boldsymbol{x}, \boldsymbol{b} \rangle = 1 \right\}$$

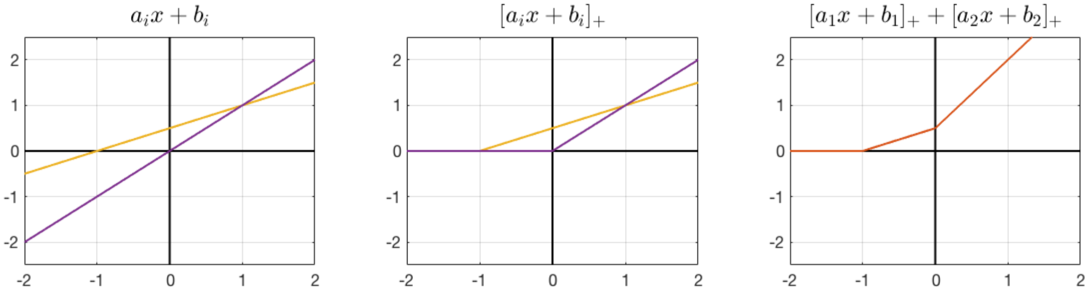
  involves solving a piece-wise linear equation.

Intuition ... (1/2)

- Consider a toy problem with $c_i = 1$ and $n = 2$:

$$\left[0, a_1 x + b_1\right]_+ + \left[0, a_2 x + b_2\right]_+ = d,$$

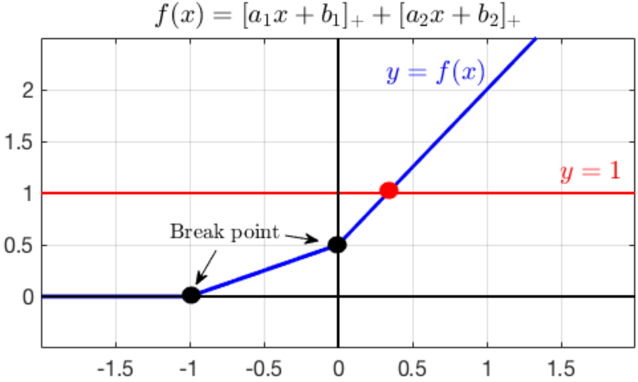
with $a_1 > 0, a_2 > 0,\ b_1 > 0$ and $b_2 = 0$.



For $d = 0$, we see that the solution set is $\{x \mid x \leq -1\}$.

Intuition ... (2/2)

For $d = 1$, solving $\left[0, a_1 x + b_1\right] + \left[0, a_2 x + b_2\right] = 1$ means find the red point:



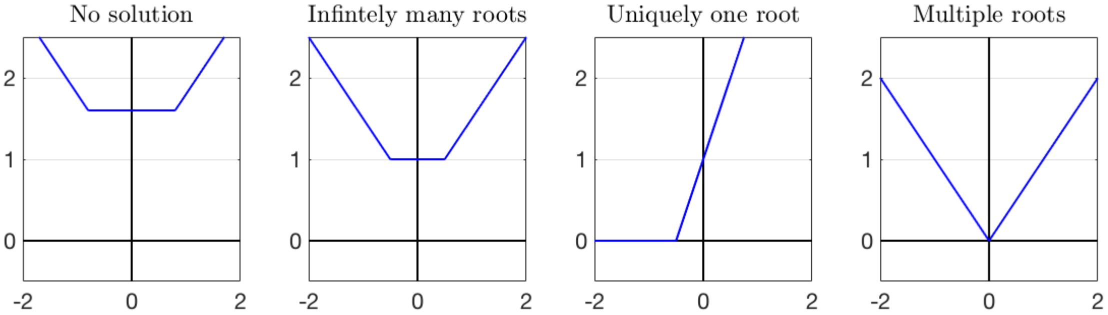
From both $d = 0$ and $d = 1$, we see that

- The piece-wise linear function f is divided into 3 line segments by 2 break points.
- Finding the root corresponds to finding the point that intersects f and the line $y = d$. The x-coordinate of the red point is the solution.
- This gives us a solution approach: find the line segment that intersects with $y = d$.

Nature of the solution

- In general $f(x) = 0$ can have $\begin{cases} \text{unique root} \\ \text{multiple roots} \\ \text{infinitely many roots} \\ \text{no solution} \end{cases}$

- For example: $[a_1 x + b_1]_+ + [a_2 x + b_2]_+ = 1,$



Solution approach

- If $d = 0$, we have

$$\sum_{i=1}^{n} c_i \big[a_i x + b_i\big]_+ = 0.$$

We find x by solving $\max\{0, a_i x + b_i\} = 0$ for all i. This is simple.

- If $d \neq 0$, we divide the equation by d and absorbing d into c_i gives

$$\sum_{i=1}^{n} c_i \big[a_i x + b_i\big]_+ = 1.$$

The remaining slides focus on this problem.

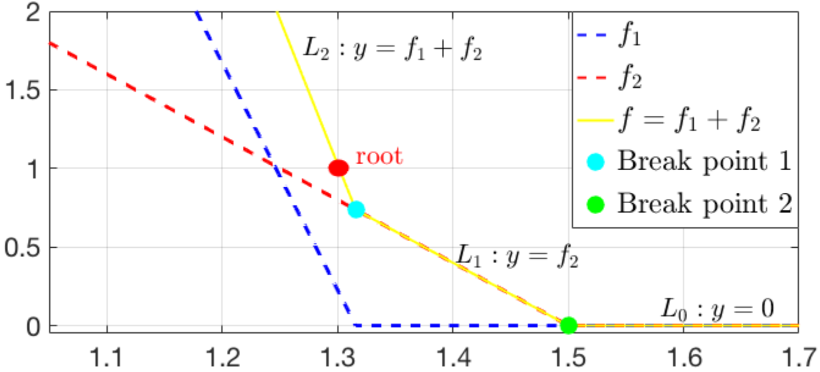
Solution approach ... (1/2)

For understanding, consider an example: $\underbrace{3.8\big[-3.8x+5\big]_+}_{f_1}+\underbrace{2\big[-2x+3\big]_+}_{f_2}=1.$

The break points (the point that changes the behavior of the function) of f_1, f_2 are:

$$BP_1=\frac{5}{3.8}=1.3158, \quad BP_2=\frac{3}{2}=1.5.$$

The break points divide f into three line segments: L_0, L_1 and L_2.



Solution approach ... (2/2)

- Firs we test L_0: does the line segment L_0 intersect with $y = 1$?
  No since $y = 0$ for all $x \in \text{dom} L_0 = \{w \mid w \geq 1.5\}$.

- Then we move to L_1: does the line segment L_1 intersect with $y = 1$?
  No since $y = f_2 < 1$ for $x \in$dom$L_1 = \{w \mid 1.3158 \leq w \leq 1.5\}$

- Then we move to L_2: does the line segment L_2 intersect with $y = 1$?
  Yes $y = f_1 + f_2$ contains 1 for $x \in$dom$L_2 = \{w \mid w \leq 1.3158\}$. And the root x is

$$\begin{array}{rrcl} & f(x) & = & 1 \\ \iff & f_1(x) + f_2(x) & = & 1 \\ \iff & 3.8[-3.8x + 5]_+ + 2[-2x + 3]_+ & = & 1 \\ \iff & 3.8(-3.8x + 5) + 2(-2x + 3) & = & 1 \\ \iff & -18.4x + 25 & = & 1 \\ \iff & x & = & 1.304. \end{array}$$

  The 3rd $\iff$ is the most important one: since f_1 and f_2 are nonnegative in domL_2, we can remove $[\ \cdot\ ]_+$.

Systematic solution approach

From the example, we see a systematic approach to solve piece-wise linear equation.

- Sort the break points: this gives us the interval for each of the line segments L_i.
- Starting from the rightmost line segment L_0 (correspond to the largest break point):
  - Compute the equation of the line segment L_i in the form of $y = mx + c$.
  - Check whether L_i intersect the line $y = 1$.
  - If L_i does not contains the solution, move to L_{i+1}.
- For the general problem $\sum_{i=1}^{n} c_i \big[a_i x + b_i\big]_+ = d$, we are given n points (a_i, b_i, c_i), hence the sorting on $\dfrac{-a_i}{b_i}$ takes in general $\mathcal{O}(n \log n)$ computational cost.
- As sorting is the most expensive step in the algorithm, the whole algorithm solving a piece-wise linear equation thus has the time complexity of $\mathcal{O}(n \log n)$.

Remarks

- Only the terms with $c_i \neq 0$ contributes to $\sum_{i=1}^{n} c_i \big[a_i x + b_i\big]_+$, ignore terms with $c_i = 0$.
- Assume there are $K \leq n$ nonzero c_i, then sorting has $\mathcal{O}(n + K \log n)$ cost.
- Therefore, in general, the cost of solving the piece-wise linear equation is in between $\mathcal{O}(n + K \log n)$ and $\mathcal{O}(n \log n)$, depending on K
- If $K \ll n$, then $n + K \log n \approx n$ and we say that the cost of solving the piece-wise linear equation is in between $\mathcal{O}(n)$ and $\mathcal{O}(n \log n)$.

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