AICE1004 MadBookPro

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"Young man, in maths you don't understand things.
You just get used to them."

von Neumann

Contents

1 Logic and set theory

1.1 Logic

1. True or False: decide whether the following sentence is a proposition
   - Two plus two equals four.
   - Two plus two equals three.
   - Two plus two is not equal to five.
   - Seven is a prime number.
   - Seven is an interesting number.
   - I am gay.
2. Give
   - Give a proposition
   - Give a proposition with AND.
   - Give a proposition with OR.
   - Give a proposition with NOT.
   - Give a proposition with IF ... THEN.
   - Give a tautology.
   - Give a contradiction.
   - (Difficult) Give a proposition with IFF.
3. If Q : Five is an even number, what is $\neg Q$?
4. Translate the statements
   - $(P \wedge \neg Q \vee \neg R) \implies S \implies T \wedge P$
   - $(P \wedge \neg Q \vee \neg R) \implies (S \implies T \wedge P)$

   to English
5. $(***)$ Given a real-world English sentence that corresponds to the logic $(P \wedge \neg Q \vee \neg R) \implies (S \implies T \wedge P)$
6. Use truth table to show that $P \implies Q$ is the same as $\neg P \vee Q$

P	Q	$\neg P$	$P \implies Q$	$(\neg P) \vee Q$
T	T			
T	F			
F	T			
F	F			

7. Translate "If roses are red and violets are blue, then I love you" to a logic statement. Find its negation. (Hint: $A \implies B$ is the same as $\neg A \vee B$).
8. Construct the truth table for $(P \wedge Q) \vee \big((\neg P) \wedge (\neg Q)\big)$

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$(\neg P) \wedge (\neg Q)$	$(P \wedge Q) \vee \big((\neg P) \wedge (\neg Q)\big)$
T	T					
T	F					
F	T					
F	F					

9. Construct the truth table for $(P \implies Q) \wedge \big(Q \implies P\big)$
10. Construct the truth table for the proposition $\big((\neg P) \vee Q\big) \wedge (\neg R)$
11. Are $(P \wedge Q) \vee R$ and $P \wedge (Q \vee R)$ equivalent? Prove or disprove it using truth table.
12. Show that $P \wedge (Q \vee R) \implies (P \wedge Q) \vee R$ using truth table.
13. Show that $\Big((P \implies Q) \wedge (Q \implies R)\Big) \implies (P \implies R)$ using truth table.
14. Show that $\Big((P \iff Q) \wedge (Q \iff R)\Big) \implies (P \iff R)$ using truth table.

15. Let n be integers. $P(n)$: "n is prime.", $E(n)$: "n is even.", $G(n)$: "$n \geq 3$."
What does $\forall n\bigg(\Big(P(n) \wedge G(n)\Big) \implies \neg E(n)\bigg)$ mean?

16. Let $p(x)$: "x is a politician" and $\ell(x)$: "x is a liar". Write down the following statement using predicate logic.
    - "All politicians are liars."
    - "Some politicians are liars."
    - "No politician is a liar."
    - "Some politicians are not liars."

17. Let c be all cities and p be all people. Let $P(p,c)$: "p lives in c."
    - What does $\exists p\, P(p, \text{Southampton})$ mean?
    - What does $\forall p \neg P(p, \text{London})$ mean?
    - What does $\forall c \exists p\, P(p,c)$ mean?
    - What does $\exists c \forall p\, P(p,c)$ mean? Is it true?

18. The following quote comes from Abraham Lincoln.

    > You can fool some of the people all of the time,
    > and all of the people some of the time,
    > but you can not fool all of the people all of the time.

    Let $F(p,t)$ be the predicate: "person p can be fooled at time t." Translate Lincoln's quote in logic symbol.

19. Consider the following argument.
I. Japanese love sushi.
II. I love sushi.
III. Therefore, I am Japanese.
What is wrong in this argument?

20. Consider the statement $S = \Big\{\forall x \ : \ x > 0\Big\}$.
    - Write S in plain English.
    - Simplify $\neg S$ and write $\neg S$ in plain English.

21. Consider the statement $S = \Big\{\forall x \ \exists y \ : \ x + y = 0\Big\}$.
    - Write S in plain English.
    - Simplify $\neg S$ and write $\neg S$ in plain English.

22. Consider the statements.
    - $Px = \{x \ : \ x \text{ is multiple of } 4\}$
    - $Qx = \{x \ : \ x \text{ is divisible by } 2\}$
    - $S = \Big\{\forall x \in \mathbb{N} \ : \ Px \implies Qx\Big\}$.

    Now
    - Write S in plain English.
    - Simplify $\neg S$ and write $\neg S$ in plain English.

23. Consider the statement "If he was famous then he would play football well".
Write down the contrapositive and negation of this statement.

24. Consider the statement "If you made an omelette then you broke eggs".
Write down the contrapositive and negation of this statement.

25. Consider the statement "If n is even than n^2 is even".
Write down the contrapositive and negation of this statement.

26. Consider the statement "$(\exists M)(\forall x \in \mathbb{R})(|x| \leq M)$"
Write down the negation of this statement.

27. Consider the statement "$(\forall x \in \mathbb{R})(\exists M)(|x| \leq M)$"
Write down the negation of this statement.

28. Prove $\boldsymbol{x} \implies y$ is equivalent to $(\neg y) \implies (\neg x)$

29. Consider the statement: "no pain, no gain". We write it as a logic statement as "no pain" $\implies$ "no gain". What is the contrapositive and converse, inverse of this statement?

30. True or false: $\neg((\neg A) \vee B) \vee C$ is equivalent to $(A \wedge (\neg B)) \vee C$

31. True or false: $\neg(\exists x \; : \; Px \wedge Qx)$ is equivalent to $\forall x \; : \; Px \implies \neg(Qx)$

32. True or false: $\neg(\forall x \; : \; Px \implies Qx)$ is equivalent to $\exists x \; : \; Px \wedge \neg(Qx)$

33. True or false: $P \implies (Q \implies P)$ is a tautology

34. True or false: $P \implies (Q \implies (R \implies P))$ is a tautology

35. True or false: $((P \implies Q) \wedge P) \implies P$ is a tautology

36. Let m, n are integers and $P(m, n)$ is "$m = n + 2$". True or false: $\forall m \exists n P(m, n)$

37. Let m, n are integers and $P(m, n)$ is "$m = 2n$". True or false: $\forall m \exists n P(m, n)$

38. True or false: $\forall x \forall y P(x, y)$ is the same as $\forall y \forall x P(x, y)$

39. True or false: $\neg\Big((\forall x)P(x)\Big) \equiv (\exists x)\Big(\neg P(x)\Big)$

40. True or false: $\neg\Big((\exists x)\neg P(x)\Big) \equiv (\forall x)\Big(P(x)\Big)$

41. Consider the statement "Let x, y be positive real numbers. If $xy \geq 20$ then $x \geq 4$ or $y > 5$".
    - State the converse of the original statement
    - State the contrapositive of the original statement
    - Give a counter example of the converse

42. Translate this into English: $(\forall x \in \mathbb{R})\Big(x \neq 0 \implies \exists y \in \mathbb{R}, xy = 1\Big)$

1.2 Logic solution

1. True or False: decide whether the following sentence is a proposition
   - Two plus two equals four. it is a proposition, it's truth value is true
   - Two plus two equals three. it is a proposition, it's truth value is false (false proposition is also a proposition)
   - Two plus two is not equal to five. it is a proposition, it's truth value is true
   - Seven is a prime number. it is a proposition, it's truth value is true
   - Seven is an interesting number. it is a proposition, it's truth value is unknown (it is person-dependent)
   - I am gay. it is a proposition, only I know it's truth value (which is false, now you know it)
2. Give
   - Give a proposition $1+1=2$
   - Give a proposition with AND. $1+1=2 \wedge 2+2=4$
   - Give a proposition with OR. $x \geq 0 \vee x \leq 0$
   - Give a proposition with NOT. 1 is not larger than 2
   - Give a proposition with IF ... THEN. If $x > 2$ then $x > 1$
   - Give a tautology. $1=1$
   - Give a contradiction. $1=0$
   - (Difficult) Give a proposition with IFF. if a,b, and c are real numbers, a+b = c+b if and only if a=c is true
3. If Q : Five is an even number, what is $\neg Q$?
   $\neg Q$: Five is not an even number. Or Five is an odd number.
4. Translate the statements
   - $(P \wedge \neg Q \vee \neg R) \implies S \implies T \wedge P$
   - $(P \wedge \neg Q \vee \neg R) \implies (S \implies T \wedge P)$

   to English
   "P and not Q or not R implies S, which in turn implies both T and P".
   "P and not Q or not R, implies that S in turn implies both T and P"
5. $(***)$ Given a real-world English sentence that corresponds to the logic $(P \wedge \neg Q \vee \neg R) \implies (S \implies T \wedge P)$

   "If it is sunny and not raining or not windy, then if we go to the park, we will see birds and it will be sunny."

   $$\begin{aligned} P &= \text{it is sunny} \\ Q &= \text{it is raining} \\ R &= \text{it is windy} \\ S &= \text{we go to the park} \\ T &= \text{we will see birds} \end{aligned}$$
6. Use truth table to show that $P \implies Q$ is the same as $\neg P \vee Q$

P	Q	$\neg P$	$P \implies Q$	$(\neg P) \vee Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T
7. Translate "If roses are red and violets are blue, then I love you" to a logic statement. Find its negation (Hint: $A \implies B$ is the same as $\neg A \vee B$).
   - R: rose are red
   - V: violets are blue
   - L: I love you
   - $(R \wedge V) \implies L$
   - $\neg\Big((R \wedge V) \implies L\Big) = \neg\Big(\neg(R \wedge V) \vee L\Big) = \neg\neg(R \wedge V) \wedge \neg L = (R \wedge V) \wedge \neg L$. This translate to "roses are red and violets are blue, and I don't love you".
8. Construct the truth table for $(P \wedge Q) \vee \big((\neg P) \wedge (\neg Q)\big)$

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$(\neg P) \wedge (\neg Q)$	$(P \wedge Q) \vee \big((\neg P) \wedge (\neg Q)\big)$
T	T	F	F	T	F	T
T	F	F	T	F	F	F
F	T	T	F	F	F	F
F	F	T	T	F	T	T

Note that the last column is the same as $P \iff Q$

9. Construct the truth table for $(P \implies Q) \wedge \big(Q \implies P\big)$

P	Q	$P \implies Q$	$Q \implies P$	$(P \implies Q) \wedge \big(Q \implies P\big)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

10. Construct the truth table for the proposition $\big((\neg P) \vee Q\big) \wedge (\neg R)$

Solution

P	Q	R	$\big((\neg P) \vee Q\big) \wedge (\neg R)$
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	1

11. Are $(P \wedge Q) \vee R$ and $P \wedge (Q \vee R)$ equivalent? Prove or disprove it using truth table.

P	Q	R	$P \wedge Q$	$Q \vee R$	$(P \wedge Q) \vee R$	$P \wedge (Q \vee R)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	F	T	T	T
T	F	F	F	F	F	T
F	T	T	F	T	T	F
F	T	F	F	T	F	F
F	F	T	F	T	T	F
F	F	F	F	F	F	F

The last two columns are the the same, so $(P \wedge Q) \vee R$ and $P \wedge (Q \vee R)$ are not equivalent

12. Show that $P \wedge (Q \vee R) \implies (P \wedge Q) \vee R$ using truth table.

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$(P \wedge Q) \vee R$	$P \wedge (Q \vee R) \implies (P \wedge Q) \vee R$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	1	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	1
0	1	1	1	0	0	1	1
0	1	0	1	0	0	0	1
0	0	1	1	0	0	1	1
0	0	0	0	0	0	0	1

13. Show that $\Big((P \implies Q) \wedge (Q \implies R)\Big) \implies (P \implies R)$ using truth table.

Let $S = \left\{\Big((P \implies Q) \wedge (Q \implies R)\Big) \implies (P \implies R)\right\}$

P	Q	R	P ⇒ Q	Q ⇒ R	(P ⇒ Q) ∧ (Q ⇒ R)	P ⇒ R	S
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

14. Show that $\Big((P \iff Q) \wedge (Q \iff R)\Big) \implies (P \iff R)$ using truth table.
Let $U = \left\{ \Big((P \iff Q) \wedge (Q \iff R)\Big) \implies (P \iff R) \right\}$

P	Q	R	P ⇔ Q	Q ⇔ R	(P ⇔ Q) ∧ (Q ⇔ R)	P ⇔ R	U
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	F	T	T
T	F	F	F	T	F	F	T
F	T	T	F	T	F	F	T
F	T	F	F	F	F	T	T
F	F	T	T	F	F	F	T
F	F	F	T	T	T	T	T

15. Let n be integers. $P(n)$: "n is prime.", $E(n)$: "n is even.", $G(n)$: "$n \geq 3$."
What does $\forall n \bigg(\Big(P(n) \wedge G(n) \Big) \implies \neg E(n) \bigg)$ mean? All primes greater than or equal to 3 are odd.

16. Let $p(x)$: "x is a politician" and $\ell(x)$: "x is a liar". Write down the following statement using predicate logic.

- "All politicians are liars." $\forall x \Big(p(x) \implies \ell(x) \Big)$
- "Some politicians are liars." $\exists x \Big(p(x) \wedge \ell(x) \Big)$
- "No politician is a liar." $\forall x \Big(p(x) \implies \neg \ell(x) \Big)$
- "Some politicians are not liars." $\exists x \Big(p(x) \wedge \ell(x) \Big)$

17. Let c be all cities and p be all people. Let $P(p, c)$: "p lives in c."

- What does $\exists p \, P(p, \text{Southampton})$ mean? Someone lives in Southampton
- What does $\forall p \, \neg P(p, \text{London})$ mean? No one lives in London
- What does $\forall c \exists p \, P(p, c)$ mean? Every city has someone lives in.
- What does $\exists c \forall p \, P(p, c)$ mean? Is it true? There is one city everyone lives in. False

18. The following quote comes from Abraham Lincoln.

> You can fool some of the people all of the time,
> and all of the people some of the time,
> but you can not fool all of the people all of the time.

Let $F(p, t)$ be the predicate: "person p can be fooled at time t." Translate Lincoln's quote in logic symbol.

$$\forall p \exists t F(p, t) \quad \wedge \quad \exists p \forall t F(p, t) \quad \wedge \quad \neg \Big(\forall p \forall t F(p, t) \Big)$$

19. Consider the following argument.
I. Japanese love sushi.
II. I love sushi.
III. Therefore, I am Japanese.
What is wrong in this argument?

We translate this argument into set statements.
I. Japanese $\in S :=$ {set of people love sushi}.
II. I $\in S$
Step III is wrong because $x \in S$ and $y \in S$ does not imply $x = y$

20. Consider the statement $S = \Big\{ \forall x \; : \; x > 0 \Big\}$.

- Write S in plain English. All number is strictly greater than zero.
- Simplify $\neg S$ and write $\neg S$ in plain English.

$$\begin{aligned} \neg S &= \neg \Big\{ \forall x \; : \; x > 0 \Big\} \\ &= \exists x \; : \; \neg \Big\{ x > 0 \Big\} \\ &= \exists x \; : \; x \leq 0 \qquad \iff \quad \text{"There is a non-positive number"} \end{aligned}$$

21. Consider the statement $S = \{\forall x \; \exists y \; : \; x + y = 0\}$.
    - Write S in plain English. "All number has a negative version of it"
    - Simplify $\neg S$ and write $\neg S$ in plain English.

$$\begin{aligned} \neg S &= \neg\{\forall x \; \exists y \; : \; x+y=0\} \\ &= \exists x \; : \; \neg\{\exists y \; : \; x+y=0\} \\ &= \exists x \; \forall y \; : \; \neg\{x+y=0\} \\ &= \exists x \; \forall y \; : \; x+y \neq 0 \qquad \iff \text{"There is a number that has no negative version of it"} \end{aligned}$$

22. Consider the statements.
    - $Px = \{x \; : \; x \text{ is multiple of } 4\}$
    - $Qx = \{x \; : \; x \text{ is divisible by } 2\}$
    - $S = \{\forall x \in \mathbb{N} \; : \; Px \implies Qx\}$.

    Now

    - Write S in plain English. "For all integer x, if x is multiple of 4 then x is divisible by 2"
    - Simplify $\neg S$ and write $\neg S$ in plain English.

$$\begin{aligned} \neg S &= \neg\{\forall x \in \mathbb{N} \; : \; Px \implies Qx\} \\ &= \exists x \in \mathbb{N} \; : \; \neg\{Px \implies Qx\} \\ &= \exists x \in \mathbb{N} \; : \; \neg\{(\neg Px) \vee Qx\} && \text{recall that } P \implies Q = (\neg P) \vee Q \\ &= \exists x \in \mathbb{N} \; : \; \neg(\neg Px) \; \wedge \; \neg Qx\} && \text{DeMorgan's law} \\ &= \exists x \in \mathbb{N} \; : \; Px \; \wedge \; \neg Qx && \iff \text{"There is an integer that is multiple of } 4 \text{ and is not divisible by 2 "} \end{aligned}$$

23. Consider the statement "If he was famous then he would play football well".
    Write down the contrapositive and negation of this statement.
    - If he doesn't play football well then he is not famous
    - He is famous and he doesn't play football well

    (Recall $x \implies y$ is the same as $\neg x \vee y$, so $\neg(x \implies y) = x \wedge \neg y$)

24. Consider the statement "If you made an omelette then you broke eggs".
    Write down the contrapositive and negation of this statement.
    - If you didn't break eggs then you didn't make an omelette.
    - You made an omlette and didn't break eggs

    (Recall $x \implies y$ is the same as $\neg x \vee y$, so $\neg(x \implies y) = x \wedge \neg y$)

25. Consider the statement "If n is even than n^2 is even".
    Write down the contrapositive and negation of this statement.
    - If n^2 is odd then n is odd
    - n is even and n^2 is odd

    (Recall $x \implies y$ is the same as $\neg x \vee y$, so $\neg(x \implies y) = x \wedge \neg y$)

26. Consider the statement "$(\exists M)(\forall x \in \mathbb{R})(|x| \leq M)$"
    Write down the negation of this statement. $(\forall M)(\exists x \in \mathbb{R})(|x| > M)$

27. Consider the statement "$(\forall x \in \mathbb{R})(\exists M)(|x| \leq M)$"
    Write down the negation of this statement. $(\exists x \in \mathbb{R})(\forall M)(|x| > M)$

28. Prove $\boldsymbol{x} \implies y$ is equivalent to $(\neg y) \implies (\neg x)$

$$\begin{aligned} \boldsymbol{x} \implies y &= (\neg \boldsymbol{x}) \vee y \\ &= y \vee (\neg \boldsymbol{x}) \\ &= (\neg\neg y) \vee (\neg \boldsymbol{x}) \\ &= (\neg y) \implies (\neg \boldsymbol{x}) \end{aligned}$$

29. Consider the statement: "no pain, no gain". We write it as a logic statement as "no pain" $\implies$ "no gain".
    What is the contrapositive of this statement?
    $\neg$"no gain" $\implies$ $\neg$ "no pain", or equivalently "gain" $\implies$ "pain"
    What is the converse of this statement?
    no gain $\implies$ no pain
    What is the inverse of this statement?
    $\neg$ no gain $\implies$ $\neg$ no pain, equivenly pain $\implies$ pain

original statement	if no pain, then no gain
contrapositive	if no (no gain), then no (no pain)
converse	if no gain, then no pain
inverse	if no (no pain), then no (no gain)

30. True or false: $\neg((\neg A) \vee B) \vee C$ is equivalent to $(A \wedge (\neg B)) \vee C$
True

31. True or false: $\neg(\exists x \;:\; Px \wedge Qx)$ is equivalent to $\forall x \;:\; Px \implies \neg(Qx)$
True

32. True or false: $\neg(\forall x \;:\; Px \implies Qx)$ is equivalent to $\exists x \;:\; Px \wedge \neg(Qx)$
True

33. True or false: $P \implies (Q \implies P)$ is a tautology
True

34. True or false: $P \implies (Q \implies (R \implies P))$ is a tautology
True

35. True or false: $((P \implies Q) \wedge P) \implies P$ is a tautology
True

36. Let m, n are integers and $P(m,n)$ is "$m = n + 2$". True or false: $\forall m \exists n P(m,n)$
True

37. Let m, n are integers and $P(m,n)$ is "$m = 2n$". True or false: $\forall m \exists n P(m,n)$
False

38. True or false: $\forall x \forall y P(x,y)$ is the same as $\forall y \forall x P(x,y)$
True

39. True or false: $\neg\Big((\forall x)P(x)\Big) \equiv (\exists x)\Big(\neg P(x)\Big)$
True

40. True or false: $\neg\Big((\exists x)\neg P(x)\Big) \equiv (\forall x)\Big(P(x)\Big)$
True

41. Consider the statement "Let x, y be positive real numbers. If $xy \geq 20$ then $x \geq 4$ or $y > 5$".

   - State the converse of the original statement

   Let x, y be positive real numbers. If $x \geq 4$ or $y > 5$ then $xy \geq 20$

   - State the contrapositive of the original statement

   Let x, y be positive real numbers. If $x < 4$ and $y \leq 5$ then $xy < 20$.

   - Give a counter example of the converse

   $$x = 4, y = 1$$

42. Translate this into English: $(\forall x \in \mathbb{R})\Big(x \neq 0 \implies \exists y \in \mathbb{R}, xy = 1\Big)$
For any real number x, if x is not zero, then there exists a real number y such that the product xy equals to 1.

1.3 Set

1. Consider three sets $\varnothing, \{0\}$ $\{\varnothing\}$ and $\{\{\}\}$, which pair of sets are equivalent?
2. $E(x)$: "x is even", $F(X)$: "x is integer ≥ 0", $G(x)$: "x is multiples of 5". Write down the elements inside the set $E(x) \cap F(x) \cap G(x)$.
3. $S = \{x \in \mathbb{N} \ : \ 1 \leq x \leq 6\}$. Write down all the element of S
4. $S = \{x \in \mathbb{N} \ : \ 1 \leq x \leq 6\}$ and $T = \{x \in \mathbb{N} : |x^2 - 6x + 1| < 8\}$. What is the relationship between S and T?
5. $S = \{a, b\}$. Is $\{a, b\} \subset S$?
6. $S = \{a, \{b\}, \{a, b\}\}$. Is $\{a, b\} \in S$? Is $\{a, b\} \subset S$?
7. $A = \{1, 2, 3, \{1, 2, 3\}\}$, $B = \{1, 2, \{1, 2\}\}$.
   Find $A \cup B$, $A \cap B$, $A \setminus B$ and $B \setminus A$.
   Explain why $A \cap B$ does not equal to $B \setminus A$.
8. If $A = \{a, b, c\}$, find 2^A.
9. If $A = \{a, \{b, c\}\}$, find 2^A.
10. If $A = \{2, 5, 10, 20\}$, find 2^A.
11. Is it true that $A \in 2^A$ holds for any set A?
12. Is it true that $A \subset 2^A$ holds for any set A?
13. Is it true that $\{A\} \in 2^A$ holds for any set A?
14. Is it true that $\{A\} \subset 2^A$ holds for any set A?
15. Let $A = \{1, 2, 3, 4\}$ and $S = \{(x, y) \ : \ x \in A, y \in A, (2 - x)(2 + y) > 2(y - x)\}$. List all elements of S.
16. If $A = \{0, 1\}, B = \{1, 2, 3\}$, find $A \times B$
17. If $A = \{0, 1\}$, find $A \times \mathbb{N}$.
18. If $A = \{0, 1\}$, can you describe what is the geometry of $A \times \mathbb{R}_+$?
19. $A = \{-1, 1\}, B = \{0, 1, 2\}$. Is $(-1, 1) \in A \times B$? Is $(0, 1) \in A \times B$?
20. If $A = \{a, b\}, B = \{2, 3\}$, $C = \{3, 4\}$.
    - Find $A \times (B \cup C)$
    - Find $A \times B$ and $A \times C$), then find $(A \times B) \cup (A \times C)$
    - Does $A \times (B \cup C) = (A \times B) \cup (A \times C)$ here?
21. $A = \{a, b, c, d\}, B = \{a, c, g\}, C = \{c, g, m, n, p\}$.
    Find $A \cap B$, $A \cap C$ and $(A \cap B) \cup (A \cap C)$.
    Does $(A \cap B) \cup (A \cap C) \stackrel{?}{=} A \cap (B \cup C)$.
22. $A = \{1, 2, 3\}, B = \{2, 5, 7\}, C = \{2, 3, 6, 7\}, D = \{2, 3\}, E = \{3\}, F = \{7, 8\}$
    - $A \cup B =?$
    - $A \cap C =?$
    - $(A \cap B \cap C) \stackrel{?}{=} E$
    - $D \stackrel{?}{\subset} A?$
    - $E \cap A =?$
    - $A \cap F =?$
    - $A \setminus B =?$
    - $D \setminus A =?$
    - $A \setminus F =?$
    - $A \Delta E =?$
    - $A \Delta C =?$
    - $D \stackrel{?}{\subset} 2^A$
    - $F \stackrel{?}{\subset} 2^{A \cup C}$
23. Prove $B \subset A = B \cap A^c$.
24. Prove $(A \setminus B) \cap B = \varnothing$.
25. Prove the De Morgan's law $(A \cup B)^c = A^c \cap B^c$.
26. Prove $A \times (B \cap C) = (A \times B) \cap (B \times C)$.
27. Is $A \subset B$ equivalent to $A \cap B = A$?
    If yes, prove it using **Theorem (Set equivalences)** in the notes. If no, give a counter example.

28. Is $A \subset B$ equivalent to $A \cup B = B$?
    If yes, prove it using **Theorem (Set equivalences)** in the notes. If no, give a counter example.

29. Is $A \subset B$ equivalent to $B^c \subset A^c$?
    If yes, prove it using **Theorem (Set equivalences)** in the notes. If no, give a counter example.

30. Is $2^A \cup 2^B$ equivalent to $2^{A \cup B}$?
    If yes, prove it using **Theorem (Set equivalences)** in the notes. If no, give a counter example.

31. Is $(A \subset B) \implies (A \subsetneq B)$ a tautology?
    If yes, prove it. If no, give a counter example.

32. True or false: if a set S is a subset of $\varnothing$, then S must be the empty set.

33. True or false: $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$

34. True or false: $A \cup (B \setminus C) = (A \cup B) \setminus (A \cup C)$

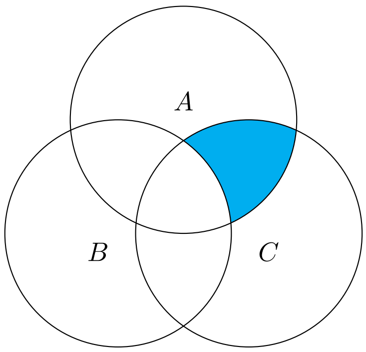
35. Picture: draw the Venn diagrams of $\overline{A \cap B}$

36. Picture: draw the Venn diagrams of $A \setminus \overline{B}$

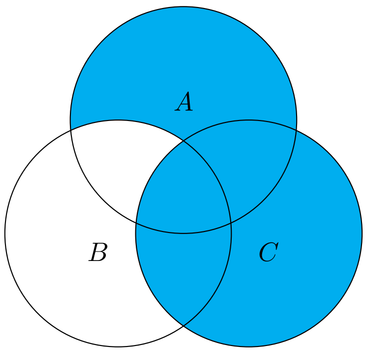
37. Picture: draw the Venn diagrams of $(A \cap C) \cup (B \cap C)$

38. Picture: draw the Venn diagrams of $(A \cap \overline{B}) \cup (A \cap C)$

39. Picture: find the equation for the Venn diagrams



40. Picture: find the equation for the Venn diagrams



41. Given $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$, find
    - $A + B$ where $+$ is Minkowski sum, i.e., $C = A + B$ is defined as $\{a + b \ : \ a \in A, b \in B\}$
    - $A - B$ where $+$ is Minkowski subtraction, i.e., $C = A - B = A + (-B)$ where $-B$ is defined as $\{-b \ : \ b \in B\}$

42. Given $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$, find
    - $A + B$ where $+$ is Minkowski sum, i.e., $C = A + B$ is defined as $\{a + b \ : \ a \in A, b \in B\}$
    - $A - B$ where $+$ is Minkowski subtraction, i.e., $C = A - B = A + (-B)$ where $-B$ is defined as $\{-b \ : \ b \in B\}$
    - $B - A$
    - Does $A - B = B - A$?

43. If $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$, find
    - $|A|$
    - $|B|$
    - $|A \times B|$

- $|A + B|$

44. Find the cardinality of the following sets
    - $A = \{\{\}, \{\}\}$
    - $B = \{\{\}, \varnothing\}$
    - $C = \{1, \{1\}, \{1, \{1\}\}\}$
    - $D = \mathbb{N}$
    - $E = \{\mathbb{N}\}$
    - $F = \{\mathbb{N}, \{\mathbb{N}\}\}$

45. Let set $k\mathbb{N}$ be the set of all elements in $\mathbb{N}$ multiplied by a number k.
    - Let $A = \mathbb{N}$ and $B = (-1)\mathbb{N}$. Write down few elements of the sets. Which set has a bigger cardinality?
    - Let $A = \mathbb{N}$ and $B = 2\mathbb{N}$. Write down few elements of the sets. Which set has a bigger cardinality?
    - Let $A = \mathbb{N}$ and $B = \{\mathbb{N}, 2\mathbb{N}\}$. Write down few elements of the sets. Which set has a bigger cardinality?
    - Let $A = \mathbb{N}$ and $B = \mathbb{N} \cap 2\mathbb{N}$. Write down few elements of the sets. Which set has a bigger cardinality?
    - Let $A = \mathbb{N}$ and $B = \mathbb{N} \cup 2\mathbb{N}$. Write down few elements of the sets. Which set has a bigger cardinality?
    - Let $A = \mathbb{N}$ and $B = \mathbb{N} + 2\mathbb{N}$ where $+$ is Minkowski sum. Write down few elements of the sets. Which set has a bigger cardinality?
    - Let $A = \mathbb{N}$ and $B = (2\mathbb{N}) \setminus \mathbb{N}$. Write down few elements of the sets. Which set has a bigger cardinality?
    - Let $A = \mathbb{N}$ and $B = (2\mathbb{N}) \times \mathbb{N}$. Write down few elements of the sets. Which set has a bigger cardinality?
    - Let $A = \mathbb{N}$ and $B = \{n^2 \ : \ n \in \mathbb{N}\}$. Write down few elements of the sets. Which set has a bigger cardinality?

1.4 Set solution

1. Consider three sets $\varnothing, \{0\}$ $\{\varnothing\}$ and $\{\{\}\}$, which pair of sets are equivalent?
   Answer: within the first three, none of them are equivalent.

   - $\varnothing$ contains nothing but both $\{0\}$ and $\{\varnothing\}$ have one element. $\varnothing \neq \{0\}$ and $\varnothing \neq \{\varnothing\}$
   - $\{0\}$ contains a number zero, $\{\varnothing\}$ contains an the empty set. A number is not the empty set. $\{0\} \neq \{\varnothing\}$

   For the last two, they are equivalent, since $\varnothing$ is just another notation of $\{\}$

2. $E(x)$: "x is even", $F(X)$: "x is integer ≥ 0", $G(x)$: "x is multiples of 5". Write down the elements inside the set $E(x) \cap F(x) \cap G(x)$.
   $$\{0, 10, 20, 30, ...\}$$

3. $S = \{x \in \mathbb{N} \ : \ 1 \leq x \leq 6\}$. Write down all the element of S
   $$S = \{1, 2, 3, 4, 5, 6\}$$

4. $S = \{x \in \mathbb{N} \ : \ 1 \leq x \leq 6\}$ and $T = \{x \in \mathbb{N} : |x^2 - 6x + 1| < 8\}$. What is the relationship between S and T?
   $$T = \{1, 2, 4, 5, 6\} \subset S$$

5. $S = \{a, b\}$. Is $\{a, b\} \subset S$? Yes because this is the same as $S \overset{?}{\subset} S$. Every set is subset of itself.

6. $S = \{a, \{b\}, \{a, b\}\}$. Is $\{a, b\} \in S$? Yes. Is $\{a, b\} \subset S$? No because $b \notin S$. Recall that if $A \subset B$ then all elements in A has to present in B

7. $A = \{1, 2, 3, \{1, 2, 3\}\}$, $B = \{1, 2, \{1, 2\}\}$. Find $A \cup B$, $A \cap B$, $A \setminus B$ and $B \setminus A$. Explain why $A \cap B$ does not equal to $B \setminus A$.
   $$A \cup B = \{1, 2, 3, \{1, 2\}, \{1, 2, 3\}\}, \quad A \cap B = \{1, 2\}, \quad A \setminus B = \{3, \{1, 2, 3\}\}, \quad B \setminus A = \{\{1, 2\}\}.$$
   $A \cap B$ does not equal to $B \setminus A$ because $A \cap B$ has two *distinct* elements, while $B \setminus A$ has one element.

8. If $A = \{a, b, c\}$, find 2^A.
   $$\left\{ \begin{matrix} \varnothing, \\ \{a\}, \{b\}, \{c\}, \\ \{a, b\}, \{a, c\}, \{b, c\}, \\ A \end{matrix} \right\}$$

9. If $A = \{a, \{b, c\}\}$, find 2^A.
   $$\left\{ \begin{matrix} \varnothing, \\ \{a\}, \{b, c\}, \\ A \end{matrix} \right\}$$

10. If $A = \{2, 5, 10, 20\}$, find 2^A.
    $$\left\{ \begin{matrix} \varnothing, \\ \{2\}, \{5\}, \{10\}, \{20\}, \\ \{2, 5\}, \{2, 10\}, \{2, 20\}, \{5, 10\}, \{5, 20\}, \{10, 20\}, \\ \{2, 5, 10\}, \{2, 5, 20\}, \{2, 10, 20\}, \{5, 10, 20\}, \\ A \end{matrix} \right\}$$

11. Is it true that $A \in 2^A$ holds for any set A?
    Yes by definition.

12. Is it true that $A \subset 2^A$ holds for any set A?
    The answer is No
    Recall the definition of subset: a set X is a subset of Y, if every element x in X, also appear in Y.
    We now give an example. Let $A = \{\{1\}\}$. Then $2^A = \{\varnothing, \{\{1\}\}\}$. Then we see that $\{1\} \notin 2^A$.

    If you do not understand: Let $A = \{\{1\}\}$. Now write $\{1\} = a$, we have $A = \{a\}$. The power set $2^A = \{\varnothing, A\} = \{\varnothing, \{a\}\}$. Now you see that $a \in A$ but $a \notin 2^A$. Do not confuse a and $\{a\}$, they are different things.

13. Is it true that $\{A\} \in 2^A$ holds for any set A?
    No. Explanation in the last question: a and $\{a\}$ are different things.

14. Is it true that $\{A\} \subset 2^A$ holds for any set A?
    Yes

15. Let $A = \{1,2,3,4\}$ and $S = \{(x,y) \ : \ x \in A, y \in A, (2-x)(2+y) > 2(y-x)\}$. List all elements of S.
First we simplify the set S. The expression $(2-x)(2+y) > 2(y-x)$ is the same as $xy < 4$. Hence

(x,y)	$< 4?$	$\in S?$
$1*1=1$	yes	yes
$1*2=2$	yes	yes
$1*3=3$	yes	yes
$1*4=4$	no	no
$2*1=2$	yes	yes
$2*2=4$	no	no
$2*3=6$	no	no
$2*4=8$	no	no
$3*1=2$	yes	yes
$3*2=6$	no	no
$3*3=9$	no	no
$3*4=12$	no	no
$4*1=4$	no	no
$4*2=8$	no	no
$4*3=12$	no	no
$4*4=16$	no	no

$$S = \Big\{(1,1),\ (1,2),\ (1,3),\ (2,1),\ (3,1)\Big\}$$

16. If $A = \{0,1\}, B = \{1,2,3\}$, find $A \times B$

$$A \times B = \Big\{(0,1),(0,2),(0,3),(1,1),(1,2),(1,3)\Big\}$$

17. If $A = \{0,1\}$, find $A \times \mathbb{N}$.

$$A \times \mathbb{N} = \left\{ \begin{array}{l} (0,0),(0,1),(0,2),(0,3),\cdots \\ (1,0),(1,1),(1,2),(1,3),\cdots \end{array} \right\}$$

18. If $A = \{0,1\}$, can you describe what is the geometry of $A \times \mathbb{R}_+$?

$$A \times \mathbb{R}_+ \ = \ \text{two parallel lines: the nonnegative x-axis and the one shifted one unit up}$$

19. $A = \{-1,1\}, B = \{0,1,2\}$. Is $(-1,1) \in A \times B$? Is $(0,1) \in A \times B$?

$$(-1,1) \in A \times B \text{ because } -1 \in A,\ 1 \in B$$

$$(0,1) \notin A \times B \text{ because } 0 \notin A \ \text{ (and then we don't need to check is } 1 \in B\text{)}$$

20. If $A = \{a,b\}, B = \{2,3\}$, $C = \{3,4\}$.

- Find $A \times (B \cup C)$
- Find $A \times B$ and $A \times C$), then find $(A \times B) \cup (A \times C)$
- Does $A \times (B \cup C) = (A \times B) \cup (A \times C)$ here?

$$\begin{array}{rcl} B \cup C &=& \{2,3,4\} \\ A \times (B \cup C) &=& \{(a,2),(a,3),(a,4),(b,2),(b,3),(b,4)\} \\ A \times B &=& \{(a,2),(a,3),(b,2),(b,3)\} \\ A \times C &=& \{(a,3),(a,4),(b,3),(b,4)\} \\ (A \times B) \cup (A \times C) &=& A \times (B \cup C) \end{array}$$

21. $A = \{a,b,c,d\}, B = \{a,c,g\}, C = \{c,g,m,n,p\}$.
Find $A \cap B$, $A \cap C$ and $(A \cap B) \cup (A \cap C)$.
Does $(A \cap B) \cup (A \cap C) \stackrel{?}{=} A \cap (B \cup C)$.

$$A \cap B = \{a,c\}, \quad A \cap C = \{c\}, \quad (A \cap B) \cup (A \cap C) = \{a,c\} = A \cap (B \cup C)$$

22. $A = \{1,2,3\}, B = \{2,5,7\}, C = \{2,3,6,7\}, D = \{2,3\}, E = \{3\}, F = \{7,8\}$

- $A \cup B = ?$ $\{1,2,3,5,7\}$
- $A \cap C = ?$ $\{2,3\}$
- $(A \cap B \cap C) \stackrel{?}{=} E$ $A \cap B \cap C = \{2\} \neq E$

- $D \overset{?}{\subset} A$? Yes
- $E \cap A =$? E
- $A \cap F =$? $\varnothing$
- $A \setminus B =$? $\{1,3\}$
- $D \setminus A =$? $\varnothing$
- $A \setminus F =$? A
- $A \Delta E =$? $\{1,2\}$
- $A \Delta C =$? $\{1,6,7\}$
- $D \overset{?}{\subset} 2^A$ yes
- $F \overset{?}{\subset} 2^{A \cup C}$ No

23. Prove $B \subset A = B \cap A^c$.

$$B \subset A \;=\; \Big\{x \;:\; x \in B \;\wedge\; x \notin A\Big\} \;=\; \Big\{x \;:\; x \in B \;\wedge\; x \in A^c\Big\} \;=\; B \cap A^c$$

24. Prove $(A \setminus B) \cap B = \varnothing$.

$$(A \setminus B) \cap B \;=\; \Big\{x \;:\; x \in (A \setminus B) \;\wedge\; x \in B\Big\} \;=\; \Big\{x \;:\; x \in A \;\wedge\; x \notin B \;\wedge\; x \in B\Big\} \;=\; \varnothing$$

since by the law of excluded middle, there is no element x satisfying both $x \notin B$ and $x \in B$.

25. Prove the De Morgan's law $(A \cup B)^c = A^c \cap B^c$.

$$(A \cup B)^c \;=\; \Big\{x \;:\; x \notin (A \;\cup\; B)\Big\} \;=\; \Big\{x \;:\; x \notin A \;\wedge\; x \notin B\Big\} \;=\; \Big\{x \;:\; x \in A^c \;\wedge\; x \in B^c\Big\} \;=\; A^c \cap B^c$$

26. Prove $A \times (B \cap C) = (A \times B) \cap (B \times C)$.

$$\begin{aligned} A \times (B \cap C) &= \Big\{(x,y) \;:\; x \in A \;\wedge\; y \in B \cap C\Big\} \\ &= \Big\{(x,y) \;:\; x \in A \;\wedge\; y \in B \;\wedge\; y \in C\Big\} \\ &= \Big\{(x,y) \;:\; (x,y) \in A \times B \;\wedge\; (x,y) \in A \times C\Big\} \\ &= (A \times B) \cap (B \times C) \end{aligned}$$

27. Is $A \subset B$ equivalent to $A \cap B = A$?
Yes. Proof: Let $C = B \setminus A$.

$$\begin{array}{rcl|l} A \cap B &=& A \cap (A \cup C) & B = A \cup C \text{ by definition} \\ &=& (A \cap A) \cup (A \cap C) & \text{distribution law} \\ &=& A \cup (A \cap C) & \text{idempotent law} \\ &=& A \cup \varnothing & \text{by definition of } C \\ &=& A & \text{identity laws} \end{array}$$

28. Is $A \subset B$ equivalent to $A \cup B = B$?
Yes. Proof:

$$A \cup B \overset{\text{commutative}}{=} B \cup A \overset{\text{by last question}}{=} B \cup (A \cap B) \overset{\text{absorption law}}{=} B$$

29. Is $A \subset B$ equivalent to $B^c \subset A^c$?
Yes. By $A \subset B$ is equivalent to $A \cap B = A$, we prove $A \cap B = A$ is equivalent to $B^c \subset A^c$.

$$\begin{array}{rcl|l} A \cap B &=& A & \\ \overline{A \cap B} &=& A^c & \text{apply complement on the whole expression} \\ A^c \cup B^c &=& A^c & \text{De Morgan's laws} \\ B^c \cup A^c &=& A^c & \text{commutative} \end{array}$$

From the last question: $X \cup Y = Y$ is equivalent to $X \subset Y$, hence $B^c \cup A^c = A^c$ gives $B^c \subset A^c$.

30. Is $2^A \cup 2^B$ equivalent to $2^{A \cup B}$?
No. $A = \{1\}, B = \{2\}$.

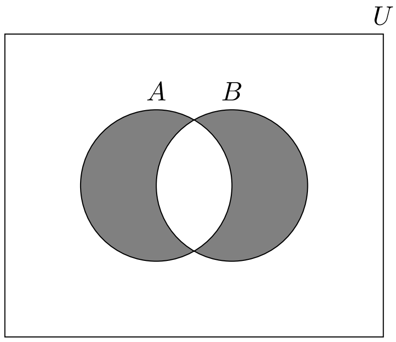
31. Is $(A \subset B) \implies (A \subsetneq B)$ a tautology?
Yes

32. True or false: if a set S is a subset of $\varnothing$, then S must be the empty set.
True

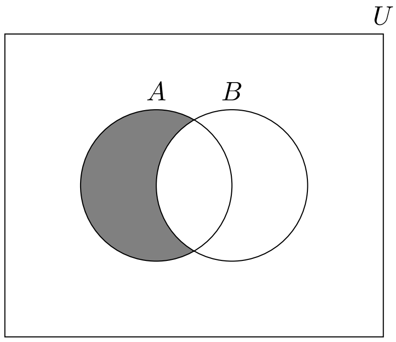
33. True or false: $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$
    True

34. True or false: $A \cup (B \setminus C) = (A \cup B) \setminus (A \cup C)$
    False. Take $C = \varnothing, A = B = \{1\}$

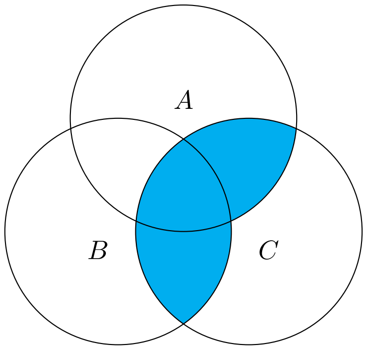
35. Picture: draw the Venn diagrams of $\overline{A \cap B}$



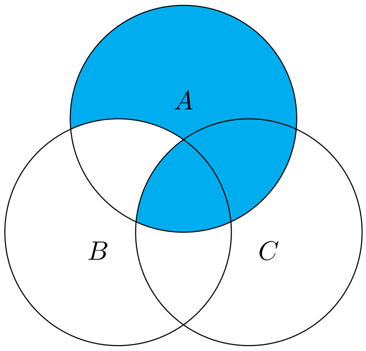
36. Picture: draw the Venn diagrams of $A \setminus \overline{B}$



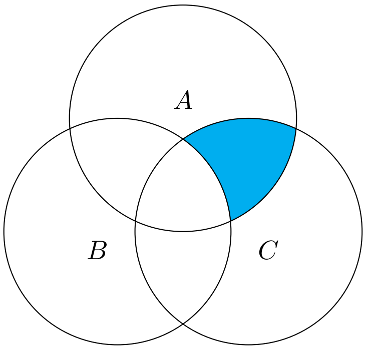
37. Picture: draw the Venn diagrams of $(A \cap C) \cup (B \cap C)$



38. Picture: draw the Venn diagrams of $(A \cap \overline{B}) \cup (A \cap C)$

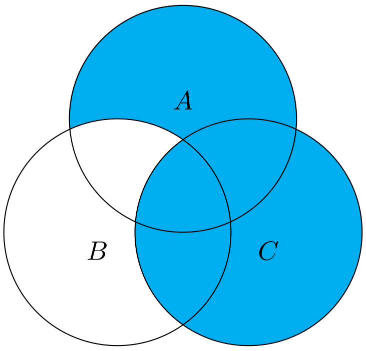


39. Picture: find the equation for the Venn diagrams



$$A \cap \overline{B} \cap C$$

40. Picture: find the equation for the Venn diagrams



$$(A \cup C) \setminus (A \cap B) \cup (A \cap B \cap C) \quad = \quad (A \cap \overline{B}) \cup C$$

41. Given $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$, find
    - $A + B$ where $+$ is Minkowski sum, i.e., $C = A + B$ is defined as $\{a + b \ : \ a \in A, b \in B\}$
      $$A + B = \Big\{a+1, a+2, a+3, b+1, b+2, b+3, c+1, c+2, c+3\Big\}$$
    - $A - B$ where $+$ is Minkowski subtraction, i.e., $C = A - B = A + (-B)$ where $-B$ is defined as $\{-b \ : \ b \in B\}$
      $$A - B = \Big\{a-1, a-2, a-3, b-1, b-2, b+3, c-1, c-2, c-3\Big\}$$

42. Given $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$, find
    - $A + B$ where $+$ is Minkowski sum, i.e., $C = A + B$ is defined as $\{a + b \ : \ a \in A, b \in B\}$
      $$\begin{aligned} A + B &= \Big\{1+2, 1+4, 1+6, 3+2, 3+4, 3+6, 5+2, 5+4, 5+6\Big\} \\ &= \Big\{3, 5, 7, 5, 7, 9, 7, 9, 11\Big\} \\ &= \Big\{3, 5, 7, 9, 11\Big\} \end{aligned}$$
    - $A - B$ where $+$ is Minkowski subtraction, i.e., $C = A - B = A + (-B)$ where $-B$ is defined as $\{-b \ : \ b \in B\}$
      $$\begin{aligned} A - B &= \Big\{1-2, 1-4, 1-6, 3-2, 3-4, 3-6, 5-2, 5-4, 5-6\Big\} \\ &= \Big\{-1, -3, -5, 1, -1, -3, 3, 1, -1\Big\} \\ &= \Big\{-5, -3, -1, 1, 3\Big\} \end{aligned}$$
    - $B - A$
      $$B - A = \Big\{-3, -1, 1, 3, 5\Big\}$$
    - $A - B \neq B - A$

43. If $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$, find
    - $|A| = 3$ because A has 3 elements
    - $|B| = 3$ because A has 3 elements
    - $|A \times B| = 9$ because it has 9 elements

- $|A+B| = |\{3,5,7,9,11\}| = 5$

44. Find the cardinality of the following sets

- $A = \{\{\},\{\}\}$
  $|A| = 1$. Set does not consider repeated elements. The two $\{\}$ are the same, so A contains one element, thus $|A| = 1$.
- $B = \{\{\},\varnothing\}$
  $|B| = 1$. $\{\}$ and $\varnothing$ are the same thing, just different symbols. Hence $B = A$.
- $C = \{1,\{1\},\{1,\{1\}\}\}$
  $|C| = 3$ because $1 \neq \{1\} \neq \{1,\{1\}\}$, they are three different things.
- $D = \mathbb{N}$
  - worst answer: ∞
  - better answer: countable infinite or $\aleph_0$
- $E = \{\mathbb{N}\}$
  $|E| = 1$. The set E contains one element, that is the set $\mathbb{N}$, here $\mathbb{N}$ is a set, but it is an element inside E.
- $F = \{\mathbb{N},\{\mathbb{N}\}\}$
  $|F| = 2$.

45. Let set $k\mathbb{N}$ be the set of all elements in $\mathbb{N}$ multiplied by a number k.

- Let $A = \mathbb{N}$ and $B = (-1)\mathbb{N}$. Write down few elements of the sets. Which set has a bigger cardinality?

$$\begin{aligned} A &= \{1,2,3,...\} \\ B &= \{-1,-2,-3,...\} \\ |A| &= \aleph_0 \\ |B| &= \aleph_0 \end{aligned}$$

  The two sets have the same cardinality, which is countable infinite.
- Let $A = \mathbb{N}$ and $B = 2\mathbb{N}$. Write down few elements of the sets. Which set has a bigger cardinality?

$$\begin{aligned} A &= \{1,2,3,...\} \\ B &= \{2,4,6,...\} \\ |A| &= \aleph_0 \\ |B| &= \aleph_0 \end{aligned}$$

  The two sets have the same cardinality, which is countable infinite.
- Let $A = \mathbb{N}$ and $B = \{\mathbb{N}, 2\mathbb{N}\}$. Write down few elements of the sets. Which set has a bigger cardinality?

$$\begin{aligned} A &= \{1,2,3,...\} \\ B &= \{\mathbb{N},2\mathbb{N}\} \\ |A| &= \aleph_0 \\ |B| &= 2 \end{aligned}$$

  A has bigger cardinality.
  Note: $\mathbb{N}$ and $2\mathbb{N}$ are different. Recall that two sets are equal if they have the same elements. $2\mathbb{N}$ only has even number, it does not have 1,3,5,... so $2\mathbb{N}$ is different from $\mathbb{N}$
- Let $A = \mathbb{N}$ and $B = \mathbb{N} \cap 2\mathbb{N}$. Write down few elements of the sets. Which set has a bigger cardinality?

$$\begin{aligned} A &= \{1,2,3,...\} \\ B &= \mathbb{N} \cap 2\mathbb{N} = 2\mathbb{N} = \{2,4,6,...\} \\ |A| &= \aleph_0 \\ |B| &= \aleph_0 \end{aligned}$$

  The two sets have the same cardinality, which is countable infinite.
- Let $A = \mathbb{N}$ and $B = \mathbb{N} \cup 2\mathbb{N}$. Write down few elements of the sets. Which set has a bigger cardinality?

$$\begin{aligned} A &= \{1,2,3,...\} \\ B &= \mathbb{N} \cup 2\mathbb{N} = \mathbb{N} = A \\ |A| &= \aleph_0 \\ |B| &= \aleph_0 \end{aligned}$$

  The two sets have the same cardinality, which is countable infinite.

- Let $A = \mathbb{N}$ and $B = \mathbb{N} + 2\mathbb{N}$ where $+$ is Minkowski sum. Write down few elements of the sets. Which set has a bigger cardinality?

$$\begin{aligned} A &= \{1,2,3,...\} \\ B &= \{n+2m \ : \ n \in \mathbb{N}, m \in \mathbb{N}\} = \{\text{all odd and even numbers greater than or equal to 3}\} \\ |A| &= \aleph_0 \\ |B| &= \aleph_0 \text{ because } B \text{ is bijective to } \mathbb{N} \end{aligned}$$

  The two sets have the same cardinality, which is countable infinite.

- Let $A = \mathbb{N}$ and $B = (2\mathbb{N}) \setminus \mathbb{N}$. Write down few elements of the sets. Which set has a bigger cardinality?

$$\begin{aligned} A &= \{1,2,3,...\} \\ B &= \{2,4,6,..\} \setminus \{1,2,3,4,...\} = \varnothing \\ |A| &= \aleph_0 \\ |B| &= 0 \end{aligned}$$

  A has bigger cardinality, , which is countable infinite.

- Let $A = \mathbb{N}$ and $B = (2\mathbb{N}) \times \mathbb{N}$. Write down few elements of the sets. Which set has a bigger cardinality?

$$\begin{aligned} A &= \{1,2,3,...\} \\ B &= \{(2n,m) \ : \ n \in \mathbb{N}, m \in \mathbb{N}\} \\ |A| &= \aleph_0 \\ |B| &= \aleph_0 \text{ by diagonal proof (see lecture notes)} \end{aligned}$$

  The two sets have the same cardinality, which is countable infinite.

- Let $A = \mathbb{N}$ and $B = \{n^2 \ : \ n \in \mathbb{N}\}$. Write down few elements of the sets. Which set has a bigger cardinality?

$$\begin{aligned} A &= \{1,2,3,...\} \\ B &= \{1,4,9,...\} \\ |A| &= \aleph_0 \\ |B| &= \aleph_0 \text{ because } B \text{ is bijective to } \mathbb{N} \end{aligned}$$

  The two sets have the same cardinality, which is countable infinite.

1.5 Relation

1. Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5\}$. Let R be a relation $<$ from A to B. List all the ordered pairs of R
2. Let a set $A = \{1, 2, 3\}$ and a relation $R = \{(1, 1), (2, 3), (3, 2)\}$ on $A \times A$.
   - What is the domain of R?
   - What is the range of R?
   - What is the inverse R^{-1}?
   - Is R reflexive?
   - Is R symmetric?
   - Is R transitive?
3. Let $A = \mathbb{N}$ and relation R defined by
$$R = \{(x, y) \ : \ x, y \in \mathbb{N}, x^3 - y^3 = 1\}.$$
   - Is R reflexive?
   - Is R symmetric?
   - Is R transitive?
   - Is R an equivalence relation?
4. Let $A = \mathbb{N}$ and relation R defined by
$$R = \{(x, y) \ : \ x, y \in \mathbb{N}, |x| = |y|\}.$$
   - Is R reflexive?
   - Is R symmetric?
   - Is R transitive?
   - Is R an equivalence relation? If yes, list the equivalence classes.
5. Let $A = \mathbb{N}$ and relation R defined by
$$R = \{(x, y) \ : \ x, y \in \mathbb{N}, x + 2y = 12\}.$$
   - List every member of R.
   - What is the domain of R?
   - What is the range of R?
   - What is the inverse R^{-1}?
6. Let $A = \{1, 2, 3, ..., 5\}$. The relation R is defined as follows: aRb if 3 divides $a - b$.
   - List every member of R.
   - What property of R satisfies?
7. Is $<$ on $\mathbb{N}$ an equivalence relation?
8. Let $A = \mathbb{N}$. A relation R on A is defined as follows: for any $x \in \mathbb{N}$ and $y \in \mathbb{N}$, we have xRy if x is a factor of y.
   - Is R reflexive?
   - Is R transitive?
   - Is R symmetric?
   - Is R an equivalence relation?
9. Let $A = \mathbb{N}$. A relation R on A is defined as follows: for any $x \in \mathbb{N}$ and $y \in \mathbb{N}$, we have xRy if $x - y$ is divisible by 12.
   - Is R reflexive?
   - Is R transitive?
   - Is R symmetric?
   - Is R an equivalence relation?
10. Let $A = \{1, 2, ..., 24\}$. Consider three subsets of A as
    - $X = \{3, 6, 9, ..., 24\}$
    - $Y = \{1, 4, 7, ...22\}$
    - $Z = \{2, 5, 8, ..., 23\}$

    - Show that X, Y, Z is a partition of A.

- If X, Y, Z are equivalence classes of A, what is the equivalence relation which yields this partition?

11. Let $A = \mathbb{N} \times \mathbb{N}$, the set of ordered pairs of positive integers (including zero). Let $\simeq$ be the relation in A defined by
$$(a, b) \simeq (c, d) \quad \text{if and only if} \quad a + d = b + c.$$
Find the equivalence class of $(2, 5)$, that is, find $[(2, 5)]$.

12. Assume $a \neq b$ in a set A with a equivalence relation R.
True or false: if $[a] \cap [b] \neq \varnothing$, then $[a] = [b]$

13. Assume R is a non-empty relation in a set A.
True or false: R is symmetric, then R^{-1} is symmetric.

14. True or false: if a set S that is partially ordered, then S must be not totally ordered

15. Let $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 5\}$ and a relation $<$ from A to B as $\{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$. Find the composition $R \circ R^{-1}$.

16. Consider the set $\mathbb{N} \times \mathbb{N}$. A relation $\sim$ in $\mathbb{N} \times \mathbb{N}$ is defined by
$$(a, b) \sim (c, d) \iff ad = bc.$$
   - Prove that $\sim$ is an equivalence relation.
   - What is the equivalence class $[(1, 1)]$?
   - Does $[(1, 1)] = [(2, 2)]$?

17. Let R, S, T be three relations on $\{1, 2, 3\} \times \{1, 2, 3\}$, defined by
$$\begin{array}{ll} (a, b)R(c, d) & \text{if} \quad a = c \\ (a, b)S(c, d) & \text{if} \quad b \leq d \\ (a, b)T(c, d) & \text{if} \quad a = c \text{ and } b \leq d \end{array}$$
Then
   - Is R anti-symmetric?
   - Is S symmetric?
   - Is S anti-symmetric?
   - Is T symmetric?

18. Let $f(x)$ be a function satisfying the relation $f(x + 1) = f(x) + f(1)$. Given $f(2) = 1$, find $f(4)$.

19. Let $X = \{1, 2, 3, 4\}$. State whether or not each of the following relations is a function from X into X.
   (a) $f = \Big\{(2, 3), (1, 4), (2, 1), (3, 2), (4, 4)\Big\}$
   (b) $g = \Big\{(3, 1), (4, 2), (1, 1)\Big\}$
   (c) $h = \Big\{(2, 1), (3, 4), (1, 4), (2, 1), (4, 4)\Big\}$

20. Let $X = \{1, 2, 3, 4, 5\}$ and
$$f = \Big\{(1, 3), (2, 5), (3, 3), (4, 1), (5, 2)\Big\}, \qquad g = \Big\{(1, 4), (2, 1), (3, 1), (4, 2), (5, 3)\Big\}$$
Find $g \circ f$ and $f \circ g$

21. Let $\mathbb{Z}$ be the set of integers. Consider the equivalence relation R on $\mathbb{Z}$ defined by xRy if $x - y = 2k$ for some $k \in \mathbb{Z}$. How many equivalence classes are in the quotient $\mathbb{Z}/R$? List these equivalence classes.

22. (Clock arithmetic) Let $\mathbb{N}$ be the set of integers $\{0, 1, 2, 3, ...\}$. Consider the equivalence relation R on $\mathbb{N}$ defined by xRy if $x \equiv y$ (mod12). How many equivalence classes are in the quotient $\mathbb{N}/R$? List these equivalence classes.

1.6 Relation solution

1. Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5\}$. Let R be a relation $<$ from A to B. List all the ordered pairs of R

$$R = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$$

2. Let a set $A = \{1, 2, 3\}$ and a relation $R = \{(1, 1), (2, 3), (3, 2)\}$ on $A \times A$.
   - What is the domain of R? $\text{dom}R = \{1, 2, 3\} = R$
   - What is the range of R? range$R = \{1, 2, 3\} = R$
   - What is the inverse R^{-1}? $R^{-1} = R$
   - Is R reflexive? no because $2 \in A$ but $(2, 2) \notin A$
   - Is R symmetric? yes because $R^{-1} = R$
   - Is R transitive? no because $(3, 2) \in R, (2, 3) \in R$ but $(3, 3) \notin R$

3. Let $A = \mathbb{N}$ and relation R defined by
$$R = \{(x, y) \ : \ x, y \in \mathbb{N}, x^3 - y^3 = 1\}.$$
   - Is R reflexive? No. $1^3 - 1^3 \neq 1$
   - Is R symmetric? No. $1^3 - 0^3 = 1$ but $0^3 - 1^3 \neq 1$
   - Is R transitive? No. $1^3 - 0^3 = 1$ and $0^3 - (-1)^3 = 1$ but $1^3 - (-1)^3 \neq 1$
   - Is R an equivalence relation? No

4. Let $A = \mathbb{N}$ and relation R defined by
$$R = \{(x, y) \ : \ x, y \in \mathbb{N}, |x| = |y|\}.$$
   - Is R reflexive? Yes, $|a| = |a|$ for all a
   - Is R symmetric? Yes, $|a| = |b|$ then $|b| = |a|$
   - Is R transitive? Yes, $|a| = |b|$ and $|b| = |c|$ then $|a| = |c|$
   - Is R an equivalence relation? If yes, list the equivalence classes. Yes. $\Big\{\{0\}, \{1, -1\}, \{2, -2\}, ...\Big\}$

5. Let $A = \mathbb{N}$ and relation R defined by
$$R = \{(x, y) \ : \ x, y \in \mathbb{N}, x + 2y = 12\}.$$
   - List every member of R. $R = \Big\{(2, 5), (4, 4), (6, 3), (8, 2), (10, 1)\Big\}$
   - What is the domain of R? $\{2, 4, 6, 8, 10\}$
   - What is the range of R? $\{1, 2, 3, 4, 5\}$
   - What is the inverse R^{-1}? $R^{-1} = \Big\{(5, 2), (4, 4), (3, 6), (2, 8), (1, 10)\Big\}$

6. Let $A = \{1, 2, 3, ..., 5\}$. The relation R is defined as follows: aRb if 3 divides $a - b$.
   - List every member of R.
   - What property of R satisfies?

$$\Big\{(1, 1), \ (1, 4), \ (2, 2), \ (2, 5), \ (3, 3), \ (4, 1), \ (4, 4), \ (5, 2), \ (5, 5)\Big\}$$

   - R is reflexive
   - R is symmetric
   - R is transitive

   R is an equivalence relation

7. Is $<$ on $\mathbb{N}$ an equivalence relation?
   No. It is not reflexive: $a < a$ is not true.

8. Let $A = \mathbb{N}$. A relation R on A is defined as follows: for any $x \in \mathbb{N}$ and $y \in \mathbb{N}$, we have xRy if x is a factor of y.
   - Is R reflexive? yes, all integers are factor of themselves
   - Is R transitive? yes
   - Is R symmetric? no
   - Is R an equivalence relation? no

9. Let $A = \mathbb{N}$. A relation R on A is defined as follows: for any $x \in \mathbb{N}$ and $y \in \mathbb{N}$, we have xRy if $x - y$ is divisible by 12.

- Is R reflexive? yes
- Is R transitive? yes
- Is R symmetric? yes
- Is R an equivalence relation? yes

10. Let $A = \{1, 2, ..., 24\}$. Consider three subsets of A as

    - $X = \{3, 6, 9, ..., 24\}$
    - $Y = \{1, 4, 7, ...22\}$
    - $Z = \{2, 5, 8, ..., 23\}$

    - Show that X, Y, Z is a partition of A.

    $$X \cap Y = \varnothing, \quad X \cap Z = \varnothing, \quad Y \cap Z = \varnothing, \quad X \cup Y \cup Z = A.$$

    - If X, Y, Z are equivalence classes of A, what is the equivalence relation which yields this partition?

    $$R = \{\text{"has the same remainder when divided by 3 as"}\}$$

11. Let $A = \mathbb{N} \times \mathbb{N}$, the set of ordered pairs of positive integers (including zero). Let $\simeq$ be the relation in A defined by

    $$(a, b) \simeq (c, d) \quad \text{if and only if} \quad a + d = b + c.$$

    Find the equivalence class of $(2, 5)$, that is, find $[(2, 5)]$.

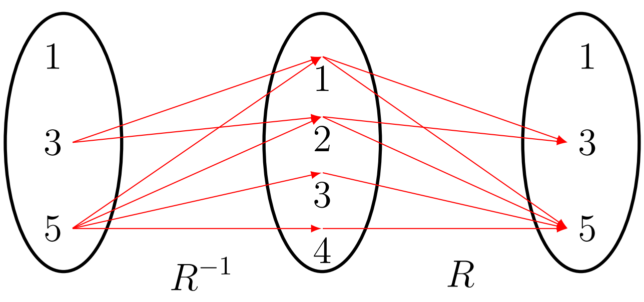
    $$[(2,5)] = \Big\{(1,4),\ (2,5),\ (3,6),\ (4,7), ..., (n, n+3), ...\Big\}$$

12. Assume $a \neq b$ in a set A with a equivalence relation R.
    True or false: if $[a] \cap [b] \neq \varnothing$, then $[a] = [b]$
    True.
    Explanation: $[a] \cap [b] \neq \varnothing$ means that there is an element e that $e \in [a]$ and $e \in [b]$. Then we have $[e] = [a]$ and $[e] = [b]$ by the uniqueness of relation. Lastly, by transitivity of equivalence relation, $[a] = [b]$.

13. Assume R is a non-empty relation in a set A.
    True or false: R is symmetric, then R^{-1} is symmetric.
    True

14. True or false: if a set S that is partially ordered, then S must be not totally ordered
    false

15. Let $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 5\}$ and a relation $<$ from A to B as $\{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$. Find the composition $R \circ R^{-1}$.
    First $R^{-1} = \{(3, 1), (5, 1), (3, 2), (5, 2), (5, 3), (5, 4)\}$.
    To find $R \circ R^{-1}$, draw the diagram (draw R^{-1} first)



$$R^{-1} \circ R = \{(3, 3), (3, 5), (5, 3), (5, 5)\}$$

16. Consider the set $\mathbb{N} \times \mathbb{N}$. A relation $\sim$ in $\mathbb{N} \times \mathbb{N}$ is defined by

    $$(a, b) \sim (c, d) \quad \Longleftrightarrow \quad ad = bc.$$

    Prove that $\sim$ is an equivalence relation.

    - For all $(a, b) \in \mathbb{N} \times \mathbb{N}$ we have $ab = ba$ hence $(a, b) \sim (a, b)$ and therefore $\sim$ is reflexive.
    - For all $(a, b) \sim (c, d)$, we have $ad = bc$ which implies $cb = da$ and therefore $(c, d) \sim (a, b)$. Thus $\sim$ is symmetric.

- To prove transitive, consider $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$. Then $ad = bc$ and $cf = de$. Thus

$$(ad)(cf) = (bc)(de).$$

Cancelling from both sides, we get $af = be$ and so $(a,b) \sim (e,f)$, and thus $\sim$ is transitive.

$\sim$ is reflective, symmetric and transitive, so it is an equivalence relation.

$$[(1,1)] = \Big\{(x,x) \mid x \in \mathbb{N}\Big\} = \Big\{(1,1),(2,2),(3,3),...\Big\}$$

$$[(1,1)] = [(2,2)]$$

17. Let R, S, T be three relations on $\{1,2,3\} \times \{1,2,3\}$, defined by

$$\begin{array}{ll} (a,b)R(c,d) & \text{if} \quad a = c \\ (a,b)S(c,d) & \text{if} \quad b \leq d \\ (a,b)T(c,d) & \text{if} \quad a = c \text{ and } b \leq d \end{array}$$

Then

- Is R anti-symmetric? — no: $(1,1)R(1,2)$ and $(1,2)R(1,1)$ but $(1,1) \neq (1,2)$
- Is S symmetric? — no: $(1,1)S(1,2)$ and $(1,2)\not{S}(1,1)$
- Is S anti-symmetric? — no: $(1,1)S(2,1)$ and $(2,1)S(1,1)$ but $(1,1) \neq (2,1)$
- Is T symmetric? — no: $(1,1)T(1,2)$ and $(1,2)\not{T}(1,1)$

18. Let $f(x)$ be a function satisfying the relation $f(x+1) = f(x) + f(1)$. Given $f(2) = 1$, find $f(4)$.

$$\begin{array}{rcl} f(2) & = & f(1) + f(1) = 1 \implies f(1) = 0.5 \\ f(3) & = & f(2) + f(1) = 1 + 0.5 = 1.5 \\ f(4) & = & f(3) + f(1) = 1.5 + 0.5 = 2 \end{array}$$

19. Let $X = \{1,2,3,4\}$. State whether or not each of the following relations is a function from X into X.

(a) $f = \Big\{(2,3),(1,4),(2,1),(3,2),(4,4)\Big\}$ — No, $(2,3),(2,1)$ means $2 \in X$ is mapped twice

(b) $g = \Big\{(3,1),(4,2),(1,1)\Big\}$ — No, $2 \in X$ is not mapped

(c) $h = \Big\{(2,1),(3,4),(1,4),(2,1),(4,4)\Big\}$ — Yes, even $(2,1) = (2,1)$ so $(2,1)$ only appeared once

20. Let $X = \{1,2,3,4,5\}$ and

$$f = \Big\{(1,3),(2,5),(3,3),(4,1),(5,2)\Big\}, \qquad g = \Big\{(1,4),(2,1),(3,1),(4,2),(5,3)\Big\}$$

Find $g \circ f$ and $f \circ g$

$$g \circ f = g(f) = \Big\{(1,1),(2,3),(3,1),(4,4),(5,1)\Big\}, \qquad f \circ g = f(g) = \Big\{(1,1),(2,3),(3,3),(4,5),(5,3)\Big\},$$

21. Let $\mathbb{Z}$ be the set of integers. Consider the equivalence relation R on $\mathbb{Z}$ defined by xRy if $x - y = 2k$ for some $k \in \mathbb{Z}$. How many equivalence classes are in the quotient $\mathbb{Z}/R$? List these equivalence classes.

Solution $\Big\{xRy \text{ if } x - y = 2k \text{ for some } k \in \mathbb{Z}\Big\}$ means x and y are related if their difference is an even number. Since the domain of R is $\mathbb{Z}$, we can say that $\Big\{xRy \text{ if } x - y = 2k \text{ for some } k \in \mathbb{Z}\Big\}$ means two integers x and y are related if their difference is an even number.

Now the relation R partitions the set of integers into two equivalence classes:

- class 1: the set of all even integers $\{..., -4, -2, 0, 2, 4, ...\}$
- class 2: the set of all odd integers $\{..., -3, -1, 1, 3, ...\}$

Each integer either falls into the category of being even or odd, and no integer can be both. Therefore, the quotient set $\mathbb{Z}/R$ has exactly two equivalence classes

22. (Clock arithmetic) Let $\mathbb{N}$ be the set of integers $\{0, 1, 2, 3, ...\}$. Consider the equivalence relation R on $\mathbb{N}$ defined by xRy if $x \equiv y \ (\text{mod} 12)$. How many equivalence classes are in the quotient $\mathbb{N}/R$? List these equivalence classes.
Solution This question is on clock arithmetic. We will call "1 o'clock", "13 o'clock" and "25 o'clock" all the same because $1 \equiv 13 \ (\text{mod} 12)$ and $1 \equiv 25 \ (\text{mod} 12)$, where $x \equiv y \ (\text{mod} 12)$ means x and y have the same remainder if divided by 12. The relation R partitions the set $\mathbb{N}$ into 12 equivalence classes

$$\begin{array}{rl} \text{class 1} & \{0, 12, 24, ...\} \\ \text{class 2} & \{1, 13, 25, ...\} \\ \vdots & \\ \text{class 12} & \{11, 23, 37, ...\} \end{array}$$

Thus the cardinality of $\mathbb{N}/R$ is 12.

1.7 (Extra) Advanced Set theory topics

Set theory is everywhere in mathematics. Here are some advanced topics on set theory from other branches of mathematics. The questions here serve two purposes: 1) they can be used as set theory problem, 2) they show you what are the "advanced topics" in mathematics look like.

1. **Point-set Topology (The foundation of geometry)**
   Given a *finite* set X. A topology on X, denoted by $\mathcal{T}$, is a subset of 2^X such that the following three conditions hold

   (a) $\varnothing$ and X belong to $\mathcal{T}$

   (b) The union of any subsets in $\mathcal{T}$ belongs to $\mathcal{T}$

   (c) The intersection of any subsets in $\mathcal{T}$ belongs to $\mathcal{T}$

   Here is an example. Let $X = \{x, y, z\}$ be a set with three elements.
   Then $2^X = \{\varnothing, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, X\}$.
   Suppose $\mathcal{T} = \{\varnothing, \{x\}, X\}$. Then $\mathcal{T}$ is a topology on X. We prove this by showing $\mathcal{T}$ fulfil the three conditions

   (a) $\mathcal{T}$ contains $\varnothing$ and X

   (b) First $\varnothing \cup \{x\} = \{x\} \in \mathcal{T}$, then $\varnothing \cup X = X \in \mathcal{T}$, lastly $\{x\} \cup X = X \in \mathcal{T}$

   (c) First $\varnothing \cap \{x\} = \varnothing \in \mathcal{T}$, then $\varnothing \cup X = \varnothing \in \mathcal{T}$, lastly $\{x\} \cap X = \{x\} \in \mathcal{T}$

   Exercise 1

   (1.1) Is $\mathcal{T} = \{\varnothing, \{x\}, \{x, y\}, X\}$ a topology on X? If yes, prove it. If no, which condition(s) is(are) violated?

   (1.2) Is $\mathcal{T} = \{\varnothing, \{x\}, \{y, z\}, X\}$ a topology on X? If yes, prove it. If no, which condition(s) is(are) violated?

   (1.3) Is $\mathcal{T} = \{\varnothing, \{x\}, \{x, y\}, \{y, z\}, X\}$ a topology on X? If yes, prove it. If no, which condition(s) is(are) violated?

   (1.4) Try, as many as possible, give all the topology $\mathcal{T}$ on X

   Exercise 2

   (2.1) Let $X = \{a, b\}$. List all possible topology $\mathcal{T}$ on X. Hints: there are four of them.

   (2.2) Let $X = \{a, b, c, d\}$. Show that $\mathcal{T} = \{\varnothing, \{a\}, \{b, c\}, \{a, b, c\}, \{a, d\}, X\}$ is a topology on X.

2. **$\boldsymbol{\sigma}$−algebra (The foundation of probability)**
   Given a *finite* set X. A $\boldsymbol{\sigma}$−algebra on X, denoted by Σ, is a subset of 2^X such that the following three conditions hold

   (a) X is in Σ

   (b) For any set A in Σ, so is $X \setminus A$

   (c) For any sets $A_1, A_2, ...$ in Σ, so is $A_1 \cup A_2 \cup ...$

   Here is an example. Let $X = \{a, b, c, d\}$ be a set with four elements. Suppose $\Sigma = \{\varnothing, \{a, b\}, \{c, d\}, X\}$. Then Σ is a $\boldsymbol{\sigma}$-algebra on X. We prove this by showing Σ fulfil the three conditions

   (a) Σ contains X by the definition of Σ

   (b) First $X \setminus \varnothing = X \in \Sigma$, then $X \setminus \{a, b\} = \{c, d\} \in \Sigma$, also $X \setminus \{c, d\} = \{a, b\} \in \Sigma$, lastly $X \setminus X = \varnothing \in \Sigma$.

   (c) We do it case by case
   - union of $\varnothing$ with any element in Σ is in Σ
   - union of $\{a, b\}$ with any element in Σ is in Σ
   - union of $\{c, d\}$ with any element in Σ is in Σ
   - union of X with any element in Σ is in Σ

   Exercises

   (1.2) Let $X = \{a, b, c\}$. Is $\Sigma = \{\varnothing, \{a\}, \{b\}, \{b, c\}, X\}$ a $\boldsymbol{\sigma}$-algebra on X? If yes, prove it. If no, which condition(s) is(are) violated?

   (1.1) Let $X = \{a, b, c\}$. Is $\Sigma = \{\varnothing, \{a\}, \{b, c\}, X\}$ a $\boldsymbol{\sigma}$-algebra on X? If yes, prove it. If no, which condition(s) is(are) violated?

   (1.3) Let $X = \{a, b, c\}$. Is $\Sigma = \{\varnothing, \{a\}, \{b\}, \{a, c\}, \{b, c\}, X\}$ a $\boldsymbol{\sigma}$-algebra on X? If yes, prove it. If no, which condition(s) is(are) violated?

3. **Filter (The foundation of order theory)**
Given a *finite nonempty* set X. A filter on X, denoted by F, is a subset of 2^X such that the following three conditions hold

(a) $X \in F$ and $\varnothing \notin F$
(b) For any set A in F that $A \subset B \subset X$, we have $B \in F$
(c) For any set $A \in F$ and $B \in F$, we have $A \cap B \in F$

Here is an example. Let $X = \{a, b, c\}$ be a set with three elements. Suppose $F = \{\{a, b\}, X\}$. Then F is a filter on X. We prove this by showing F fulfil the three conditions

(a) By the definition of F, we have F contains X and F not containing $\varnothing$
(b) First $\{a, b\} \in F$ and $\{a, b\} \subset X$, we have $\{a, b\} \subset X \in F$. Next $X \in F$, we have $X \subset X \in F$.
(c) From $\{a, b\} \in F$ and $X \in F$, we have $\{a, b\} \cap X = \{a, b\} \in F$

Exercises

(1.1) Let $X = \{a, b, c\}$. Is $F = \{\{a\}, \{a, b\}, \{a, c\}, X\}$ a filter on X? If yes, prove it. If no, which condition(s) is(are) violated?
(1.2) Let $X = \{a, b, c\}$. Given an example of a filter F on X.
(1.3) Let $X = \{a, b, c\}$. Given an example of not a filter F on X.
(1.4) Let $X = \{a, b\}$. Show all possible filters on X.

4. **Group (The foundation of algebra)**
Given a *non-empty* set G and an binary operation $\star$. If the following four conditions are true, then the ordered pair $(G, \star)$ is called a group.

(a) (Closure) For all a, b in G, we have $a \star b \in G$
(b) (Associative) For all a, b, c in G, we have $a \star (b \star c) = (a \star b) \star c$
(c) (Identity) There is an element $e \in G$ such that $a \star e = e \star a = a$ for all a in G
(d) (Inverse) For all a in G, there is an element $b \in G$ such that $a \star b = b \star a = e$.

Here is an example. Let $X = \mathbb{Z}$ the set of all integer and the operation $\star$ is the addition $+$. Then $(\mathbb{Z}, +)$ is a group. We prove this by showing $(\mathbb{Z}, +)$ fulfil the four conditions

(a) (Closure) Addition of any two integers a, b, we have $a + b$ being also an integer.
(b) (Associative) For all integers a, b, c, we have $a + (b + c) = (a + b) + c$.
(c) (Identity) There is an element $0 \in \mathbb{Z}$ such that $a + 0 = 0 + a = a$ for all integer a.
(d) (Inverse) For all integer a in G, there is an integer b such that $a + b = b + a = 0$. Actually we write $b = -a$.

Here are two examples of not a group.

(a) $(\mathbb{Z}, -)$ is not a group. It violates the associativity. For example $3 - (2 - 1) = 2 \neq (3 - 2) - 1 = 0$.
(b) $(\mathbb{R}, \times)$ is not a group. First we can see that 1 is the identity: $a \times 1 = 1 \times a = a$ for all $a \in \mathbb{R}$. Now, for $0 \in R$, there is no $b \in \mathbb{R}$ such that $0 \times b = b \times 0 = 1$.

Exercises

(1.1) Consider the set $\mathbb{C}$. Is $(\mathbb{C}, +)$ a group?
(1.2) Consider the set $\mathbb{R} \setminus \{0\}$. Is $\big(\mathbb{R} \setminus \{0\}, \times\big)$ a group?
(1.3) Let $X = \{1, 2, 3, 4, 6, 12\}$. Let $gcd(a, b)$ denotes the greatest common divisor of a and b. Is $(X, \gcd)$ a group?
(1.4) Let X be all 2-by-2 matrix that is invertible. Let $\cdot$ denotes matrix multiplication. Is $(X, \cdot)$ a group?

5. **Matroid (The foundation of graph theory)**
Given a finite nonempty set G (called ground set) and a set $I \subset 2^G$, the ordered pair (G, I) is a matroid if

(a) $\varnothing \in I$
(b) For each $B \subset A \subset G$, if $A \in I$ then $B \in I$
(c) If A, B are two independent sets (i.e., $A \cap B = \varnothing$) and $|A| > |B|$ (i.e., A has more elements than B), then there exists $x \in A \setminus B$ such that $B \cup \{x\}$ is in I

1.8 Proofs

1. (A taste of algebraic number theory) Prove $\sqrt{2}+\sqrt{6}<\sqrt{15}$
2. Prove by contraction, that there is no integers a and b that $18a+6b=1$.
3. Prove by contraction, that x is real and positive, then $x+\dfrac{4}{x}\geq 4$
4. (AM-GM inequality) Prove by contraction, that if a,b are positive and real, then $\dfrac{a+b}{2}\geq\sqrt{ab}$
5. Prove by contraction, that for all real θ, $\sin\theta+\cos\theta\leq\sqrt{2}$
6. (Archimedean property) Prove that there is no "the largest rational number".

1.9 Proofs solution

1. (A taste of algebraic number theory) Prove $\sqrt{2}+\sqrt{6}<\sqrt{15}$.
   To prove $a<b$ we can consider proving $a^2<b^2$.
   Why: $()^2$ removes $\surd$
   But note that to use $a<b \iff a^2<b^2$ we need to make sure $a,b>0$
   We have that since $\sqrt{2}>\sqrt{0}$ and $\sqrt{6}>\sqrt{0}$ so $\sqrt{2}+\sqrt{6}>0$
   Now by high-school math $(a+b)^2=a^2+2ab+b^2$, we have

$$\begin{aligned} \text{prove } \sqrt{2}+\sqrt{6}<\sqrt{15} &\iff \text{prove } 2+2\sqrt{12}+6<15 \\ &\iff \text{prove } 2\sqrt{12}<7 \\ &\iff \text{prove } (2\sqrt{12})^2<7^2 \\ &\iff \text{prove } 4\cdot 12<49 \end{aligned}$$

   So now we write everything backward and pretend we are math genius

$$\begin{aligned} 48<49 &\iff 4\cdot 12<49 \\ &\iff 2\sqrt{12}<7 \\ &\iff 2+2\sqrt{12}+6<15 \\ &\iff \sqrt{2}^2+2\sqrt{2\cdot 6}+\sqrt{6}^2<15 \\ &\iff (\sqrt{2}+\sqrt{6})^2<15 \\ &\iff \sqrt{2}+\sqrt{6}<\sqrt{15} \end{aligned}$$

   Lesson learned: square up a few times to eliminate the square roots.

2. Prove by contraction, that there is no integers a and b that $18a+6b=1$.
   The statement is $\forall a,b\in\mathbb{Z}, 18a+6b\neq 1$. The negation of this statement is $\exists a,b\in\mathbb{Z}, 18a+6b=1$.
   For contradiction, suppose there are integers a,b that $18a+6b=1$.
   Now we need to create a contradiction: divide $18a+6b=1$ by 6 gives

$$3a+b=\frac{1}{6} \tag{1}$$

   Here is the contradiction, because a,b are integers, so $3a+b$ is also an integer, but the equality (1) is saying that "an integer equals to an non-integer", so the assumption is false, the negation of the assumption is true, which is what we want to prove.

3. Prove by contraction, that x is real and positive, then $x+\dfrac{4}{x}\geq 4$
   The statement is $\forall x\geq 0, x+\dfrac{4}{x}\geq 4$. The negation of this statement is $\exists x\geq 0, x+\dfrac{4}{x}<4$.
   $x+\dfrac{4}{x}\leq 4 \iff x^2+4<4x$ by multiplying both side by x. We can do so because $x\geq 0$ and multiplication by a positive number does not change the sign of the inequality.
   Now rearrange $x^2+4<4x$ give $x^2-4x+4\leq 0 \iff (x-2)^2<0$.
   Contradiction: only imaginary number have squares smaller than zero, so $x-2$ must be imaginary, but we assumed x is real (so $x-2$ is also real). So the assumption is false, the negation of the assumption is true, which is what we want to prove.

4. (AM-GM inequality) Prove by contraction, that if a,b are positive and real, then $\dfrac{a+b}{2}\geq\sqrt{ab}$
   The statement is $\forall a,b\geq 0, \dfrac{a+b}{2}\geq\sqrt{ab}$. The negation of this statement is $\exists a,b\geq 0, \dfrac{a+b}{2}<\sqrt{ab}$.
   Since $a,b\geq 0$ so $0\leq\dfrac{a+b}{2}<\sqrt{ab}$ and if we take square on both side we do not change the inequality sign. Now squaring both sides gives $0\leq\dfrac{(a+b)^2}{4}<ab \iff (a+b)^2<4ab \iff a^2+2ab+b^2<4ab \iff a^2-2ab+b^2<0 \iff (a-b)^2<0$.
   Contradiction: only imaginary number have squares smaller than zero, so $a-b$ must be imaginary, but we assumed a,b are positive real. So the assumption is false, the negation of the assumption is true, which is what we want to prove.

5. Prove by contraction, that for all real θ, $\sin\theta+\cos\theta\leq\sqrt{2}$
   The statement is $\forall\theta\in\mathbb{R}, \sin\theta+\cos\theta\leq\sqrt{2}$. The negation of this statement is $\exists\theta\in\mathbb{R}, \sin\theta+\cos\theta>\sqrt{2}$.
   Squaring both sides $(\sin\theta+\cos\theta)^2>2 \iff \underbrace{\sin^2\theta+\cos^2\theta}_{1}+\underbrace{2\sin\theta\cos\theta}_{\sin 2\theta}>2 \iff \sin 2\theta>1$.
   sine of whatever angle can never be larger than 1, so contradiction.

6. (Archimedean property) Prove that there is no "the largest rational number".
   Solution in lecture notes.

1.10 Mathematical induction

1. Let $n \in \mathbb{N}$, prove $\displaystyle\sum_{i=1}^{n} i^1 = \frac{1}{2}\Big(n^2 + \frac{2}{2}n\Big)$ $\qquad 1 + 2 + \cdots + n = \frac{1}{2}\Big(n^2 + \frac{2}{2}n\Big)$

2. Let $n \in \mathbb{N}$, prove $\displaystyle\sum_{i=1}^{n} i^2 = \frac{1}{3}\Big(n^3 + \frac{3}{2}n^2 + \frac{3}{6}n\Big)$ $\qquad 1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}\Big(n^3 + \frac{3}{2}n^2 + \frac{3}{6}n\Big)$

3. Let $n \in \mathbb{N}$, prove $\displaystyle\sum_{i=1}^{n} i^3 = \frac{1}{4}\Big(n^4 + \frac{4}{2}n^3 + \frac{6}{6}n^2\Big)$ $\qquad 1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}\Big(n^4 + \frac{4}{2}n^3 + \frac{6}{6}n^2\Big)$

4. Let $n \in \mathbb{N}$, prove $\displaystyle\sum_{i=1}^{n} i^4 = \frac{1}{5}\Big(n^5 + \frac{5}{2}n^4 + \frac{10}{6}n^3 + 0n^2 - \frac{5}{30}n\Big)$ $\qquad 1^4 + 2^4 + \cdots + n^4 = \frac{1}{5}\Big(n^5 + \frac{5}{2}n^4 + \frac{10}{6}n^3 - \frac{5}{30}n\Big)$
   (If you are interested to know more, google "Faulhaber's formula")

5. Let $n \in \mathbb{N}$, prove $\displaystyle\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1.$ $\qquad 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$

6. Let $n \in \mathbb{N}$ and $n \geq 2$, prove $(n+1)! > 2^n$.

7. Let $n \in \mathbb{N}$, prove $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$ $\qquad \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

8. Let $n \in \mathbb{N}$, prove $\displaystyle\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}.$ $\qquad \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

9. Let $n \in \mathbb{N}$, prove $\displaystyle\sum_{i=1}^{n} \frac{i+2}{i(i+1)(2^i)} = 1 - \frac{1}{(n+1)2^n}.$ $\qquad \frac{3}{1 \cdot 2 \cdot 2} + \frac{4}{2 \cdot 3 \cdot 2^2} + \cdots + \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(n+1)2^n}$

10. Let $n \in \mathbb{N}$ and $n \geq 2$, prove $\displaystyle\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}.$ $\qquad \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$

11. Let $n \in \mathbb{N}$, prove $\displaystyle\prod_{i=1}^{n} \frac{2i-1}{2i} < \frac{1}{\sqrt{2n+1}}$ $\qquad \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$

12. Let $n \in \mathbb{N}$, prove $x^n - 1$ is divisible by $x - 1$.
    Hint: a polynomial $p(x)$ is divisible by a polynomial $q(x)$ if there is a polynomial $r(x)$ such that $p(x)$ can be written as $q(x)r(x)$.

13. Let $n \in \mathbb{N}$, prove $n^3 + 2n$ is divisible by 3.

14. Let $n \in \mathbb{N}$, prove $17n^3 + 103n$ is divisible by 6.

15. Let $n \in \mathbb{N}$, prove $5^{2n+1} + 2^{2n+1}$ is divisible by 7.

16. Let $n \in \mathbb{N}$ that is odd, prove $2^n + 1$ is divisible by 3.

17. Let $n \in \mathbb{N}$ that is odd, prove $n^2 - 1$ is divisible by 8.

18. Let $n \in \mathbb{N}$ that is odd, prove $n^4 - 1$ is divisible by 16.

19. Let $n \in \mathbb{N}$ and $r \neq 1$, prove $\displaystyle\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}$

20. Let $n \in \mathbb{N}$, prove $S(n) = 1 + 3 + 5 + \cdots + (2n-1) = n^2$. That is, the sum of the first n odd numbers is n^2.

21. Let $n \in \mathbb{N}$, prove the sum $S(n) = 8 + 13 + 18 + 23 + \cdots + (3 + 5n)$ has a closed-form formula $2.5n^2 + 5.5n$.

22. Let $n \in \mathbb{N}$ and $1 + x > 0$, prove $(1+x)^n \geq 1 + nx$.

23. Let $n \in \mathbb{N}$ and $n \geq 2$, prove $n^2 + 4n < 4^n$.

24. Let $n \in \mathbb{N}$, prove $\displaystyle\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} < 1.$

25. Let $n \in \mathbb{N}$, prove $\sqrt{1 + \sqrt{2 + \sqrt{3 + \cdots + \sqrt{n}}}} < 3.$

26. Let $n \in \mathbb{N}$ that is larger than 4 (4 included), prove $n! > 2^n$.

27. Let $n \in \mathbb{N}$ that is larger than 2 (2 included), prove $\displaystyle\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right) = \frac{n+1}{2n}$

28. Let $n \in \mathbb{N}$ that is larger than 2 (2 included), prove $n^{1/n} < 2 - \dfrac{1}{n}$

29. Let $n \in \mathbb{N}$. Prove the binomial theorem using induction.

30. Let $n \in \mathbb{N}$, $a \in \mathbb{R}$ and let $|a|$ denotes the absolute value of a. Given the fact that $|a+b| \leq |a| + |b|$ for any $a, b \in \mathbb{R}$, prove that

$$\left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i|$$

31. Let $n \in \mathbb{N}$. Prove that any $2^n \times 2^n$ chessboard with any one square removed can always be covered by a L-shape tiles with 3 boxes.

1.11 Mathematical induction solution

1. Let $n \in \mathbb{N}$, prove $\displaystyle\sum_{i=1}^{n} i^1 = \frac{1}{2}\Big(n^2 + \frac{2}{2}n\Big)$

   (a) Base case: $n = 1$.
   LHS (Left hand side) $= 1$
   RHS (right hand side) $\frac{1}{2}(1^2 + 1) = 1$
   LHS = RHS so base case is true.

   (b) Induction hypothesis: assume $n = k$ is true for some $k \in \mathbb{N}$

   $$\sum_{i=1}^{k} i = \frac{1}{2}\Big(k^2 + \frac{2}{2}k\Big) \qquad \text{(H)}$$

   (c) Case $n = k + 1$

   $$\begin{aligned} LHS = \sum_{i=1}^{k+1} i \;=\; \Big(\sum_{i=1}^{k} i\Big) + k + 1 &\overset{(H)}{=} \frac{1}{2}\Big(k^2 + \frac{2}{2}k\Big) + k + 1 \\ &= \frac{1}{2}\Big(k^2 + \frac{2}{2}k + 2k + 2\Big) \\ &= \frac{1}{2}\Big(k^2 + 2k + 1 + \frac{2}{2}k + 1\Big) \\ &= \frac{1}{2}\Big((k+1)^2 + \frac{2}{2}(k+1)\Big) = RHS \end{aligned}$$

   So the case for $n = k + 1$ is true.

2. Let $n \in \mathbb{N}$, prove $\displaystyle\sum_{i=1}^{n} i^2 = \frac{1}{3}\Big(n^3 + \frac{3}{2}n^2 + \frac{3}{6}n\Big)$

   (a) Base case $n = 1$: LHS $= 1^2 = 1 = \frac{1}{3}\cdot 3 = \frac{1}{3}\Big(1 + \frac{3}{2} + \frac{3}{6}\Big) = \frac{1}{3}\Big(1^3 + \frac{3}{2}1^2 + \frac{3}{6}1\Big) =$ RHS

   (b) Induction hypothesis: assume $n = k$ is true for some $k \in \mathbb{N}$

   $$\sum_{i=1}^{k} i^2 = \frac{1}{3}\Big(k^3 + \frac{3}{2}k^2 + \frac{3}{6}k\Big) \qquad \text{(H)}$$

   (c) Case $n = k + 1$

   $$\begin{aligned} LHS &= \sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 \overset{(H)}{=} \frac{1}{3}\Big(k^3 + \frac{3}{2}k^2 + \frac{3}{6}k\Big) + k^2 + 2k + 1 = \frac{1}{3}\Big(k^3 + \frac{3}{2}k^2 + \frac{3}{6}k + 3k^2 + 6k + 3\Big) \\ RHS &= \frac{1}{3}\Big((k+1)^3 + \frac{3}{2}(k+1)^2 + \frac{3}{6}(k+1)\Big) \end{aligned}$$

   We show LHS = RHS by showing LHS - RHS = 0.

   $$\begin{aligned} 3LHS - 3RHS &= \Big(k^3 + \frac{3}{2}k^2 + \frac{3}{6}k + 3k^2 + 6k + 3\Big) - \Big((k+1)^3 + \frac{3}{2}(k+1)^2 + \frac{3}{6}(k+1)\Big) \\ &= \Big(k^3 + \frac{8}{2}k^2 + \frac{39}{6}k + 3\Big) - \Big((k^3 + 3k^2 + 3k + 1) + \frac{3}{2}(k^2 + 2k + 1) + \frac{3}{6}(k+1)\Big) \\ &= 0 \end{aligned}$$

   So the case for $n = k + 1$ is true.

3. Let $n \in \mathbb{N}$, prove $\displaystyle\sum_{i=1}^{n} i^3 = \frac{1}{4}\Big(n^4 + \frac{4}{2}n^3 + \frac{6}{6}n^2\Big)$

   (a) Base case $n = 1$: LHS$= 1^3 = 1 = \frac{1}{4}\cdot 4 = \frac{1}{4}\Big(1^4 + \frac{4}{2}1^3 + \frac{6}{6}1^2\Big)$ =RHS

   (b) Induction hypothesis: assume $n = k$ is true for some $k \in \mathbb{N}$

   $$\sum_{i=1}^{k} i^3 = \frac{1}{4}\Big(k^4 + \frac{4}{2}k^3 + \frac{6}{6}k^2\Big) \qquad \text{(H)}$$

   (c) Case $n = k + 1$

   $$\begin{aligned} LHS &= \sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3 \overset{(H)}{=} \frac{1}{4}\Big(k^4 + \frac{4}{2}k^3 + \frac{6}{6}k^2\Big) + (k+1)^3 \\ RHS &= \frac{1}{4}\Big((k+1)^4 + \frac{4}{2}(k+1)^3 + \frac{6}{6}(k+1)^2\Big) \end{aligned}$$

   We show LHS = RHS by showing LHS - RHS = 0.
   Work out the details of $4LHS - 4RHS = 0$ (do it yourself) shows the case for $n = k + 1$ is true.

4. Let $n \in \mathbb{N}$, prove $\displaystyle\sum_{i=1}^{n} i^4 = \frac{1}{5}\Big(n^5 + \frac{5}{2}n^4 + \frac{10}{6}n^3 + 0n^2 - \frac{5}{30}n\Big)$

   (a) Base case $n=1$: LHS$= 1^4 = 1 = \frac{1}{5}\cdot 5 = \frac{1}{5}\Big(1^5 + \frac{5}{2}1^4 + \frac{10}{6}1^3 + 0\cdot 1^2 - \frac{5}{30}\cdot 1\Big)$ =RHS

   (b) Induction hypothesis: assume $n=k$ is true for some $k \in \mathbb{N}$

$$\sum_{i=1}^{k} i^4 = \frac{1}{5}\Big(k^5 + \frac{5}{2}k^4 + \frac{10}{6}k^3 + 0k^2 - \frac{5}{30}k\Big) \tag{H}$$

   (c) Case $n = k+1$

$$\begin{aligned} LHS &= \sum_{i=1}^{k+1} i^4 = \sum_{i=1}^{k} i^4 + (k+1)^4 \overset{(H)}{=} \frac{1}{5}\Big(k^5 + \frac{5}{2}k^4 + \frac{10}{6}k^3 + 0k^2 - \frac{5}{30}k\Big) + (k+1)^4 \\ RHS &= \frac{1}{5}\Big((k+1)^5 + \frac{5}{2}(k+1)^4 + \frac{10}{6}(k+1)^3 + 0(k+1)^2 - \frac{5}{30}(k+1)\Big) \end{aligned}$$

   We show LHS = RHS by showing LHS - RHS $=0$.
   Work out the details of $5LHS - 5RHS = 0$ (do it yourself) shows the case for $n=k+1$ is true.

5. Base case $1 \cdot 1! = 2! - 1$.

   Induction hypothesis (H): assume $\displaystyle\sum_{i=1}^{k} i \cdot i! = (k+1)! - 1$.

   Case $n = k+1$, we have

$$\begin{aligned} \sum_{i=1}^{k+1} i \cdot i! = \sum_{i=1}^{k} i\cdot i! + (k+1)\cdot(k+1)! &\overset{H}{=} (k+1)! - 1 + (k+1)\cdot(k+1)! \\ &= \Big(1 + (k+1)\Big)(k+1)! - 1 \\ &= (k+2)! - 1 \end{aligned}$$

6. Base case: for $n=2$ we have $(2+1)! = 3! = 6 > 2^2 = 4$.
   Induction hypothesis: Assume that for some $k \geq 2$,

$$(k+1)! > 2^k.$$

   Starting from the inductive hypothesis, we multiply both sides by $k+2$:

$$(k+2)\cdot(k+1)! > (k+2)\cdot 2^k.$$

   This simplifies to:

$$(k+2)! > (k+2)\cdot 2^k.$$

   Since $k+2 \geq 4$ for $k \geq 2$, we have:

$$(k+2)\cdot 2^k \geq 4\cdot 2^k = 2^{k+2}.$$

   Therefore,

$$(k+2)! > 2^{k+1}.$$

7. Base case: for $n=1$ we have $\dfrac{1}{1(2)} = \dfrac{1}{2} = \dfrac{1}{1+1}$

   Induction hypothesis (H): assume $\displaystyle\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$.

   At case $n=k+1$, we have

$$\begin{aligned} \sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} \\ &\overset{(H)}{=} \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2} \end{aligned}$$

8. Base case: for $n=1$ we have $\dfrac{1}{1(3)} = \dfrac{1}{3} = \dfrac{1}{2(1)+1}$

   Induction hypothesis (H): assume $\displaystyle\sum_{i=1}^{k} \frac{1}{(2i-1)(2i+1)} = \frac{k}{2k+1}$.

At case $n = k+1$, we have

$$\begin{aligned}\sum_{i=1}^{k+1}\frac{1}{(2i-1)(2i+1)} &= \sum_{i=1}^{k}\frac{1}{(2i-1)(2i+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)}\\ &\overset{(H)}{=} \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}\\ &= \frac{1}{2k+1}\Big(k+\frac{1}{2k+3}\Big) = \frac{1}{2k+1}\frac{2k^2+3k+1}{2k+3} = \frac{1}{2k+1}\frac{(2k+1)(k+1)}{2k+3} = \frac{k+1}{2(k+1)+1}\end{aligned}$$

9. Base case: for $n=1$, LHS$=\dfrac{3}{1\cdot 2\cdot 2} = 1-\dfrac{1}{2\cdot 2^1}$ =RHS
   Induction hypothesis:

$$\sum_{i=1}^{k}\frac{i+2}{i(i+1)(2^i)} = 1-\frac{1}{(k+1)2^k}. \tag{H}$$

Case $n=k+1$

$$\begin{aligned}\sum_{i=1}^{k+1}\frac{i+2}{i(i+1)(2^i)} &= \sum_{i=1}^{k}\frac{i+2}{i(i+1)(2^i)} + \frac{k+1+2}{(k+1)(k+1+1)(2^{k+1})}\\ &\overset{H}{=} 1-\frac{1}{(k+1)2^k} + \frac{k+3}{(k+1)(k+2)(2^{k+1})}\\ &= 1+\frac{1}{(k+1)2^k}\Big(-1+\frac{k+3}{(k+2)2}\Big)\\ &= 1+\frac{1}{(k+1)2^k}\Big(\frac{-2(k+2)}{2(k+2)}+\frac{k+3}{2(k+2)}\Big)\\ &= 1+\frac{1}{(k+1)2^k}\frac{-2(k+2)+k+3}{2(k+2)}\\ &= 1+\frac{1}{(k+1)2^{k+1}}\frac{-2k-4+k+3}{k+2}\\ &= 1+\frac{1}{(k+1)2^{k+1}}\frac{-k-1}{k+2}\\ &= 1-\frac{1}{(k+2)2^{k+1}}\end{aligned}$$

10. Base case: for $n=2$, we want to show $1+\dfrac{1}{\sqrt{2}} > \sqrt{2}$.

$$\begin{aligned}\iff\quad \sqrt{2}+1 &> 2 && \text{multiply the whole expression by } \sqrt{2}\\ \iff\quad \sqrt{2} &> 1 && \text{subtract 1 in the whole expression}\\ \iff\quad \sqrt{2} &> \sqrt{1} && +1=\sqrt{1}\\ \iff\quad 2 &> 1 && \sqrt{a}>\sqrt{b} \iff a>b \text{ if } a>0, b>0\end{aligned}$$

All the inequalities are if and only if, so $1+\dfrac{1}{\sqrt{2}} > \sqrt{2}$ is true.

Induction hypothesis: assume $\displaystyle\sum_{i=1}^{k}\frac{1}{\sqrt{i}} > \sqrt{k}$ for some $k>2$. (H)

At case $n=k+1$, we have LHS$=\displaystyle\sum_{i=1}^{k+1}\frac{1}{\sqrt{i}} = \sum_{i=1}^{k}\frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} \overset{(H)}{>} \sqrt{k}+\frac{1}{\sqrt{k+1}}$.

Now we want to show $\sqrt{k}+\dfrac{1}{\sqrt{k+1}} > \sqrt{k+1}$

$$\begin{aligned}\iff\quad \sqrt{k(k+1)}+1 &> k+1 && \text{multiply the whole expression by } \sqrt{k+1}\\ \iff\quad \sqrt{k(k+1)} &> k && \text{subtract 1 in the whole expression}\\ \iff\quad \sqrt{k^2+k} &> \sqrt{k^2}\\ \iff\quad k^2+k &> k^2 && \sqrt{a}>\sqrt{b} \iff a>b \text{ if } a>0, b>0\text{, this is true because } k>2\\ \iff\quad k &> 0 && k>0\end{aligned}$$

All the inequalities are if and only if, so $\sqrt{k}+\dfrac{1}{\sqrt{k+1}} > \sqrt{k+1}$ is true.

11. Base case: for $n=1$, we have $\frac{1}{2} \overset{?}{<} \frac{1}{\sqrt{3}}$. This is true because $2=\sqrt{4}$ and $4>3 \implies \sqrt{4}>\sqrt{3} \implies \frac{1}{\sqrt{4}} < \frac{1}{\sqrt{3}}$.

    Induction hypothesis: assume $\displaystyle\prod_{i=1}^{k} \frac{2i-1}{2i} < \frac{1}{\sqrt{2k+1}}$ is true for some $k>1$. (H)

    At case $n=k+1$, we have LHS= $\displaystyle\prod_{i=1}^{k+1} \frac{2i-1}{2i} = \prod_{i=1}^{k} \frac{2i-1}{2i} \cdot \frac{2(k+1)-1}{2(k+1)} \overset{(H)}{<} \frac{1}{\sqrt{2k+1}} \cdot \frac{2(k+1)-1}{2(k+1)}$.

    We want to show $\frac{1}{\sqrt{2k+1}} \cdot \frac{2(k+1)-1}{2(k+1)} < \frac{1}{\sqrt{2(k+1)+1}}$

$$\begin{aligned}
\iff \quad \sqrt{2(k+1)+1}\Big(2(k+1)-1\Big) &\leq 2(k+1)\sqrt{2k+1} \\
\iff \quad \sqrt{2k+3}(2k+1) &\leq 2(k+1)\sqrt{2k+1} \\
\iff \quad (2k+3)(2k+1)^2 &\leq 4(k+1)^2(2k+1) \\
\iff \quad (2k+3)(2k+1)^2 &\leq 4(k+1)^2(2k+1) \\
\iff \quad (2k+3)(4k^2+4k+1) &\leq (4k^2+8k+4)(2k+1) \\
\iff \quad 8k^3+8k^2+2k+12k^2+12k+3 &\leq 8k^3+16k^2+8k+4k^2+8k+4 \\
\iff \quad 8k^3+20k^2+14k+3 &\leq 8k^3+20k^2+16k+4 \\
\iff \quad 0 &\leq 2k+1
\end{aligned}$$

    All the inequalities are if and only if, so what we want to show is true.

12. Base case: for $n=1$ we have $x-1$ divisible by $x-1$.
    Induction hypothesis: assume x^k-1 is divisible by $x-1$. That is, we have

$$x^k - 1 = (x-1)q(x) \qquad \text{(H)}$$

    for some polynomial $q(x)$.

    At case $n=k+1$, we have

$$\begin{aligned}
x^{k+1}-1 &= x^{k+1}-(x-1)-1+(x-1) \\
&= x^{k+1}-x+1-1+x-1 \\
&= x(x^k-1)+x-1 \\
&\overset{H}{=} x(x-1)q(x)+x-1 \\
&= (xq(x)+1)(x-1) \\
&= q'(x)(x-1)
\end{aligned}$$

    Therefore, $x^{k+1}-1$ is divisible by $x-1$.

13. Base case: for $n=0$ we have 0 divisible by 3.
    Induction hypothesis: assume k^3+2k is divisible by 3. That is, we have

$$k^3+2k = 3q \qquad \text{(H)}$$

    for some polynomial $q(x)$.

    At case $n=k+1$, we have $(k+1)^3+2(k+1)$, so

$$\begin{aligned}
(k+1)^3+2(k+1) &= k^3+3k^2+3k+1+2k+2 \\
&= k^3+2k+3k^2+3k+3 \\
&= k^3+2k+3(k^2+k+1) \\
&\overset{(H)}{=} 3q+3(k^2+k+1) \\
&= 3(q+k^2+k+1)
\end{aligned}$$

    Therefore $(k+1)^3+2(k+1)$ is divisible by 3.

14. Base case: for $n=0$ we have $17\cdot 1^3+103\cdot 1 = 120 = 6\cdot 20$
    Induction hypothesis: assume for some k, the term $17k^3+103k$ is divisible by 6. That is, we have

$$17k^3+103k = 6q \qquad \text{(H)}$$

    for some positive integer q.

    At case $n=k+1$, we have $17(k+1)^3+103(k+1)$, so

$$\begin{aligned}
17(k+1)^3+103(k+1) &= 17(k^3+3k^2+3k+1)+103k+103 \\
&= 17k^3+103k+17\cdot 3k^2+17\cdot 3k+17+103 \\
&\overset{(H)}{=} 6q+3\cdot 17k^2+3\cdot 17k+120 \\
&= 6q+3(17k^2+17k)+6\cdot 20 \\
&= 6(q+20)+3\cdot 17k(k+1)
\end{aligned}$$

Note that $k(k+1)$ is even, that is, $k(k+1)$ can be written as $2m$ for some positive integer m. Therefore we have $3\cdot 17k(k+1) = 3\cdot 17\cdot 2m = 6\cdot 17m$ and thus $6(q+20)+3\cdot 17k(k+1) = 6(q+20)+6\cdot 17m = 6(q+20+17m)$. Therefore $17(k+1)^3+103(k+1)$ is divisible by 6.

15. Base case: for $n=0$ we have $5^{2\cdot 0+1}+2^{2\cdot 0+1} = 7 = 7\cdot 1$
Induction hypothesis: assume for some k, the term $5^{2k+1}+2^{2k+1}$ is divisible by 7. That is, we have

$$5^{2k+1}+2^{2k+1} = 7q \tag{H}$$

for some positive integer q.
At case $n=k+1$, we have $5^{2(k+1)+1}+2^{2(k+1)+1} = 5^{2k+2+1}+2^{2k+2+1} = 5^{2k+1+2}+2^{2k+1+2} = 5^{2k+1}\cdot 25+2^{2k+1}\cdot 4$, so

$$\begin{aligned}
5^{2(k+1)+1}+2^{2(k+1)+1} &= 5^{2k+1}\cdot(21+4)+2^{2k+1}\cdot 4\\
&= 5^{2k+1}\cdot 21+5^{2k+1}\cdot 4+2^{2k+1}\cdot 4\\
&= 5^{2k+1}\cdot 21+(5^{2k+1}+2^{2k+1})4\\
&\overset{(H)}{=} 5^{2k+1}\cdot 21+(7q)4\\
&= 5^{2k+1}\cdot 3\cdot 7+(7q)4\\
&= (5^{2k+1}\cdot 3+4q)\cdot 7
\end{aligned}$$

So $5^{2(k+1)+1}+2^{2(k+1)+1}$ is divisible by 7.

16. Base case: for $n=1$ we have $2^1+1=3$ is divisible by 3.
Induction hypothesis: assume for some odd k, the term 2^k+1 is divisible by 8. That is, we have

$$2^k+1 = 3q \tag{H}$$

for some positive integer q.
At case $n=k+2$ (not $k+1$ because k is odd and the next odd number is $k+2$), we have

$$2^{k+2}+1 \;=\; 2^k4+1 \;=\; 2^k4+1+3-3 \;=\; 2^k4+4-3 \;=\; (2^k+1)4-3 \;\overset{H}{=}\; 3q4-3 \;=\; 3(4q-1)$$

So $2^{k+2}+1$ is divisible by 3.

17. Base case: for $n=1$ we have $1^2-1=0$ is divisible by 8 (zero is divisible by everything).
Induction hypothesis: assume for some odd k, the term k^2-1 is divisible by 8. That is, we have

$$k^2-1 = 8q \tag{H}$$

for some positive integer q.
At case $n=k+2$ (not $k+1$ because k is odd and the next odd number is $k+2$), we have

$$(k+2)^2-1 \;=\; k^2+4k+4-1 \;=\; k^2-1+4(k+1) \;\overset{H}{=}\; 8q+4(k+1)$$

Since $k>1$ is odd, then $k+1$ is even. Therefore, $k+1$ can be written as $2m$ where m is an positive integer. Hence

$$(k+2)^2-1 \;=\; 8q+4(k+1) \;=\; 8q+4(2m) \;=\; 8q+8m \;=\; 8(q+m)$$

So $(k+2)^2$ is divisible by 8.

18. Base case: $1^4-1=0$ is divisible by 16
Induction hypothesis: assume for some odd k, the term k^4-1 is divisible by 16. That is, we have

$$k^4-1 = 16q \tag{H}$$

for some positive integer q.
At case $n=k+2$ (not $k+1$ because k is odd and the next odd number is $k+2$), we have

$$\begin{aligned}
(k+2)^4-1 &= k^4+4k^32+6k^24+4k8+16-1\\
&= k^4-1+8k^3+24k^2+32k+16\\
&\overset{H}{=} 16q+8k^3+24k^2+16\cdot 2k+16\\
&= 16q+8(k^3+3k^2+4k+2)\\
&= 16q+8(k+1)(k^2+2k+2)
\end{aligned}$$

Since $k>1$ is odd, then $k+1$ is even. Therefore, $k+1$ can be written as $2m$ where m is an positive integer. Hence

$$(k+2)^4-1 \;=\; 16q+8(2m)(k^2+2k+2) \;=\; 16(q+m(k^2+2k+2))$$

So $(k+2)^4-1$ is divisible by 16.

19. Base case $n = 1$, $\displaystyle\sum_{i=0}^{1} r^i = 1 + r = \frac{1-r^2}{1-r}$

    Induction hypothesis (H): assume $\displaystyle\sum_{i=0}^{k} r^i = \frac{1-r^{k+1}}{1-r}$

    At case $n = k+1$, we have

$$\begin{aligned}\sum_{i=0}^{k+1} r^i = \sum_{i=0}^{k} r^i + r^{k+1} &\overset{H}{=} \frac{1-r^{k+1}}{1-r} + r^{k+1} \\ &= \frac{1-r^{k+1}}{1-r} + r^{k+1}\frac{1-r}{1-r} \\ &= \frac{1-r^{k+1}+r^{k+1}-r^{k+2}}{1-r} = \frac{1-r^{k+2}}{1-r}\end{aligned}$$

20. Base case: $n = 1$ we have $S(1) = 1 = 1^2$.

    Induction hypothesis (H): assume $S(k) = 1 + 3 + 5 + \cdots + (2k-1) = k^2$.

    For $n = k+1$ we have

$$\begin{aligned} S(k+1) &= 1 + 3 + 5 + \cdots + (2k-1) + 2(k+1) - 1 \\ &= S(k) + 2k + 1 \\ &\overset{H}{=} k^2 + 2k + 1 \\ &= (k+1)^2 \end{aligned}$$

    Solution approach two Consider the first $m = 2n$ intergers

$$\begin{aligned} T(n) = 1 + 2 + 3 + \cdots + m &= (1 + 3 + 5 + \cdots + 2n - 1) + (2 + 4 + 6 + \cdots + 2n) \\ \frac{1}{2}\big(m^2 + m\big) &= S(n) + 2(1 + 2 + 3 + \cdots + n) \\ &= S(n) + n^2 + n \end{aligned}$$

    So

$$S(n) = \frac{1}{2}\big(m^2+m\big) - n^2 - n \overset{m=2n}{\Longrightarrow} S(n) = \frac{1}{2}\big(4n^2+2n\big) - n^2 - n = n^2.$$

21. Base case: $n = 1$ we have $S(1) = 3 + 5 = 8 = 2.5 + 5.5$.

    Induction hypothesis (H): assume $S(k) = 8 + 13 + 18 + 23 + \ldots + (3 + 5n) = 2.5k^2 + 5.5k$.

    For $n = k+1$ we have

$$\begin{aligned} S(k+1) = 8 + 13 + 18 + 23 + \ldots + (3+5k) + (3 + 5(k+1))) &= S(k) + 3 + 5k + 5 \\ &\overset{H}{=} 2.5k^2 + 5.5k + 8 + 5k \\ &= 2.5k^2 + 5.5k + 2.5 + 5.5 + 2 \times 2.5k \\ &= 2.5k^2 + 2 \times 2.5k + 2.5 + (5.5k + 5.5) \\ &= 2.5(k^2 + 2k + 1) + 5.5(k+1) \\ &= 2.5(k+1)^2 + 5.5(k+1) \end{aligned}$$

22. Base case: $n = 1$ we have $1 + x \geq 1 + x$ because $\geq$ means $>$ OR $=$.

    Induction hypothesis (H): assume $(1+x)^k \geq 1 + kx$

    For $n = k+1$, we have

$$\begin{aligned} (1+x)^{k+1} = (1+x)(1+x)^k &\overset{H}{\geq} (1+x)(1+kx) \\ &= 1 + kx + x + kx^2 \\ &= 1 + (k+1)x + kx^2 \\ &\geq 1 + (k+1)x \qquad \because k \geq 1, x^2 \geq 0 \end{aligned}$$

23. Base case: $n = 2$ we have $2^2 + 4(2) = 12 < 4^2 = 16$.

    Induction hypothesis (H): assume $k^2 + 4k < 4^k$ for some $k > 2$.

    For $n = k+1$, we want to show $(1+k)^2 + 4(k+1) < 4^{k+1}$, now (this is tricky), consider

$$\frac{(1+k)^2 + 4(k+1)}{k^2+4k} = \frac{1 + 2k + k^2 + 4k + 4}{k^2 + 4k} = \frac{k^2 + 4k + 2k + 5}{k^2+4k} = 1 + \frac{2k+5}{k^2+4k}$$

    Now for $k > 2$ we have $2k < k^2$ and $5 < 4k$ so $\dfrac{2k+5}{k^2+4k} < 1$ and therefore

$$\frac{(1+k)^2 + 4(k+1)}{k^2+4k} = 1 + \frac{2k+5}{k^2+4k} < 1 + 1 < 4$$

    Hence $(1+k)^2 + 4(k+1) < 4(k^2+4k) \overset{H}{<} 4 \cdot 4^k = 4^{k+1}$

24. Base case is true $\frac{1}{2} < 1$.

    Induction hypothesis (H): assume $\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} < 1$

    For $n = k+1$, we need some trick.

    First we look at what will not work.

    If we consider $\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}}$, then by (H) we have

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} < 1 + \frac{1}{2^{k+1}}.$$

    This approach will not work.

    Here to show the case $n = k+1$, we need a clever way to use (H), and here is the trick: consider

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} \;=\; \frac{1}{2}\Big(1 + \frac{1}{2^2} + \cdots + \frac{1}{2^{k-1}} + \frac{1}{2^k}\Big) \overset{H}{<} \frac{1}{2}\Big(1\Big) \;=\; 1.$$

25. Base case: $\sqrt{1} < 3$.

    Induction hypothesis (H): assume $\sqrt{1 + \sqrt{2 + \sqrt{3 + \cdots + \sqrt{k}}}} < 3$.

    For case $n = k+1$, we have

$$\sqrt{1 + \sqrt{2 + \sqrt{3 + \cdots + \sqrt{k + \sqrt{k+1}}}}}$$

    Now we shift all the indices $\{2, 3, ..., k, k+1\}$ by 1 by using $\ell = k+1$, we have

$$\sqrt{1 + \underbrace{\sqrt{1 + \sqrt{2 + \cdots + \sqrt{\ell - 1 + \sqrt{\ell}}}}}_{\ell = k+1 \text{ for indicies } 2,3,\ldots,k+1}} \;\overset{H}{<}\; \sqrt{1+3} = 2 < 3.$$

26. Base case: $4! = 4(3)(2)(1) = 24 > 16 = 2^4$.

    Induction hypothesis (H): assume $k! > 2^k$.

    For case $n = k+1$, we have

$$\begin{aligned} (k+1)! \;=\; (k+1)k &\overset{H}{>} 2^k \cdot k \\ &\overset{k>2}{>} 2^k \cdot 2 \\ &= 2^{k+1} \end{aligned}$$

27. Base case: $1 - \frac{1}{2^2} = \frac{3}{4} = \frac{2+1}{2(2)}$.

    Induction hypothesis (H): assume $\prod_{i=2}^{k}\Big(1 - \frac{1}{i^2}\Big) = \frac{k+1}{2k}$

    For case $n = k+1$, we have

$$\begin{aligned} \prod_{i=2}^{k+1}\Big(1 - \frac{1}{i^2}\Big) \;=\; \Big(1 - \frac{1}{(k+1)^2}\Big)\prod_{i=2}^{k}\Big(1 - \frac{1}{i^2}\Big) &\overset{H}{=} \Big(1 - \frac{1}{(k+1)^2}\Big)\frac{k+1}{2k} \\ &= \frac{(k+1)^2 - 1}{(k+1)^2} \cdot \frac{k+1}{2k} \\ &= \frac{k^2 + 2k}{2k(k+1)} \;=\; \frac{k+2}{2(k+1)} \end{aligned}$$

28. Base case: $n = 2$. We have LHS $= \sqrt{2}$ and RHS $= 2 - \frac{1}{2}$ both being positive. To show LHS $<$ RHS, we consider showing LHS^2 <RHS^2. LHS$^2 = 2$, and RHS$^2 = 4 - 2 + \frac{1}{4}$. Clearly $2 < 2 + \frac{1}{4}$ since $\frac{1}{4} > 0$.

    Induction hypothesis (H): assume $k^{1/k} < 2 - \frac{1}{k}$ for some integer $k > 2$.

For case $n = k+1$, LHS$= (k+1)^{\frac{1}{k+1}}$ and RHS$= 2 - \dfrac{1}{k+1}$. Now we do some analysis, suppose LHS<RHS, then

$$\begin{array}{rrcl}
 & (k+1)^{\frac{1}{k+1}} & < & 2 - \dfrac{1}{k+1} \\
\iff & k+1 & < & (2 - \dfrac{1}{k+1})^{k+1} \\
\iff & k & < & (2 - \dfrac{1}{k+1})^{k+1} - 1 \\
\iff & k^{1/k} & < & \Big((2 - \dfrac{1}{k+1})^{k+1} - 1\Big)^{1/k} \\
\overset{(H)}{\iff} & 2 - \dfrac{1}{k} & < & \Big((2 - \dfrac{1}{k+1})^{k+1} - 1\Big)^{1/k} \\
\iff & (2 - \dfrac{1}{k})^k & < & (2 - \dfrac{1}{k+1})^{k+1} - 1
\end{array}$$

So if we can show the last inequality is true, we proved the case for $n = k+1$.
Now we consider the following function

$$f(x) = \Big(2 - \frac{1}{x+1}\Big)^{x+1} - \Big(2 - \frac{1}{x}\Big)^x - 1$$

By derivative test, $f(x) > 0$ so the statement is true.

29. Let $n \in \mathbb{N}$. Prove the binomial theorem using induction.

Base case Consider case $n = 1$, we have $(x+y)^1 = x + y \; = \; \binom{1}{0}x^0y^1 + \binom{1}{1}x^1y^0$.
Hypothesis step Assume the statement at $n = m$

$$(x+y)^m \;=\; \sum_{k=0}^{m}\binom{m}{k}x^ky^{m-k} \tag{$\mathcal{H}$}$$

Induction step Consider the statement at $m+1$, we have

$$\begin{array}{rl}
 & (x+y)^{m+1} \\
= & (x+y)(x+y)^m \\
\overset{(\mathcal{H})}{=} & (x+y)\displaystyle\sum_{k=0}^{m}\binom{m}{k}x^ky^{m-k} \\
= & (x+y)\Big(\dbinom{m}{0}x^0y^m + \dbinom{m}{1}x^1y^{m-1} + \cdots + \dbinom{m}{m-1}x^{m-1}y^1 + \dbinom{m}{m}x^my^0\Big) \\
= & \begin{array}{l} \dbinom{m}{0}x^1y^m + \dbinom{m}{1}x^2y^{m-1} + \cdots + \dbinom{m}{m-1}x^my^1 + \dbinom{m}{m}x^{m+1}y^0 \\ +\dbinom{m}{0}x^0y^{m+1} + \dbinom{m}{1}x^1y^m + \cdots + \dbinom{m}{m-1}x^{m-1}y^2 + \dbinom{m}{m}x^my^1 \end{array} \\
= & \dbinom{m}{0}x^0y^{m+1} + \Big(\dbinom{m}{0} + \dbinom{m}{1}\Big)x^1y^m + \Big(\dbinom{m}{1} + \dbinom{m}{2}\Big)x^2y^{m-1} + \cdots + \Big(\dbinom{m}{m-1} + \dbinom{m}{m}\Big)x^my^1 + \dbinom{m}{m}x^{m+1}y^0 \\
= & \dbinom{m+1}{0}x^0y^{m+1} + \dbinom{m+1}{1}x^1y^m + \dbinom{m+1}{2}x^2y^{m-1} + \cdots + \dbinom{m}{m}x^my^1 + \dbinom{m+1}{m+1}x^{m+1}y^0 \\
= & \displaystyle\sum_{k=0}^{m+1}\binom{m+1}{k}x^ky^{m+1-k} \quad \square
\end{array}$$

30. Let $n \in \mathbb{N}$, $a \in \mathbb{R}$ and let $|a|$ denotes the absolute value of a. Given the fact that $|a+b| \leq |a| + |b|$ for any $a, b \in \mathbb{R}$, prove that

$$\left|\sum_{i=1}^{n} a_i\right| \;\leq\; \sum_{i=1}^{n}|a_i|$$

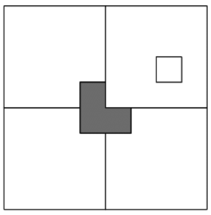
Base case For $n = 1$, we have $|a_1 + a_2| \leq |a_1| + |a_2|$ by the triangle inequality of absolute value (given in the question).
Hypothesis step Now assume the statement is true for $n = k$.

$$\left|\sum_{i=1}^{k} a_i\right| \;\leq\; \sum_{i=1}^{k}|a_i|$$

Induction step

$$\begin{aligned}
\left|\sum_{i=1}^{k+1} a_i\right| &= \left|\sum_{i=1}^{k} a_i + a_{k+1}\right| && \text{split the summation sign}\\
&\leq \left|\sum_{i=1}^{k} a_i\right| + |a_{k+1}| && \text{triangle inequality (the base case)}\\
&\leq \sum_{i=1}^{k} |a_i| + |a_{k+1}| && \text{by the induction hypothesis}\\
&= \sum_{i=1}^{k+1} |a_i| \quad \square
\end{aligned}$$

31. Let $n \in \mathbb{N}$. Prove that any $2^n \times 2^n$ chessboard with any one square removed can always be covered by a L-shape tile with 3 boxes.
Base case For $n = 1$, we have a 2-by-2 grid, removing any one square from this grid gives a L-shaped tile. So the statement is true for $n = 1$.
Hypothesis step Now assume the statement is true for $n = k$.
Induction step Now consider the case $n = k + 1$. We have a $2^{k+1} \times 2^{k+1}$ chessboard.
To prove the statement, we have one square removed from this $2^{k+1} \times 2^{k+1}$ chessboard.
We now divide the chessboard into 4 regions

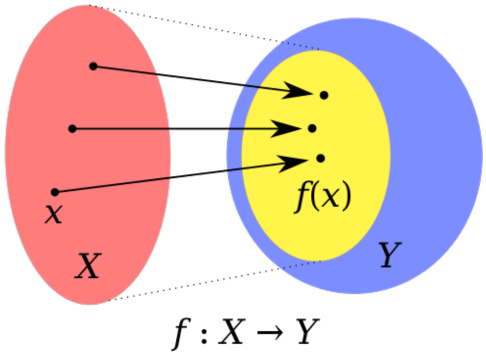


- The 4 regions are all $2^k \times 2^k$ chessboard
- We take out a L-shape tile (black in the figure) in the 3 regions opposite to the one tile removed in the whole chessboard
- By the hypothesis, each of the $2^k \times 2^k$ chessboard can be filled by the L-shaped tiles.
- Lastly, we fill the black L-shape tile, we have filled the whole $2^{k+1} \times 2^{k+1}$ chessboard by L-shape tiles

2 Calculus

2.1 Function

Revision. A function f maps from a set X to a set Y.



- The set X is called domain, we write $\mathrm{dom} f$
- The set Y is called codomain, we write $\mathrm{codom} f$
- The set $\{y \in Y | \exists x \in X \text{ s.t. } y = f(x)\}$ is called range or image.
- Common careless mistake: if $f : \mathbb{R} \to \mathbb{R}$, it is not necessary $\mathrm{dom} f$ is $\mathbb{R}$. Domain is a set that "every element in this set has mapped by f to a value in other set". It is possible some value in $\mathbb{R}$ maps to nothing.

1. Let $\mathcal{A} = \{0, 1, 2, 3\}, \mathcal{B} = \{1, 3, 5, 7\}$.
   - Write the set $\mathcal{A} \times \mathcal{B}$
   - Let $f : \mathcal{A} \to \mathcal{B}$ given by $f(x) = 2x + 1$. Is f injective? Is f surjective? Is f bijective?
2. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Let $f : X \to Y$ as follows: $f(a) = y, f(c) = x$. Is f a function?
3. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Let $f : X \to Y$ as follows: $f(a) = y, f(b) = z, f(c) = x, f(c) = z$. Is f a function?
4. Find the domain and range of $f(x) = 9 - x^2$
5. Find the domain and range of $f(x) = \dfrac{3}{3 - x}$
6. Find the domain and range of $f(x) = \dfrac{x - 2}{x + 4}$
7. Find the domain of $f(x) = \sqrt{\dfrac{1 + x}{1 - x}}$ if the range of f is restricted to $\mathbb{R}$
8. Find the domain and range of $f(x) = 5 + \sqrt{16 - x}$
9. Find the domain and range of $f(x) = -\sqrt{81 - x^2}$
10. Find the domain and range of $f(x) = x + |x| + 2$
11. Find the domain and range of $f(x) = 3\cos(4x)$
12. Let $X = [-2, 2], Y = [-1, 4]$ and $f : X \to Y$ defined as $f(x) = x^2$.
    - Is f injective?
    - Is f surjective?
    - Is f bijective?
13. Let $X = [-1, 1]$ and $f : X \to X$ defined as $f(x) = \sin x$ (we take radian, not degree).
    - Is f injective?
    - Is f surjective?
    - Is f bijective?
14. Let $X = [-1, 1]$ and $f : X \to X$ defined as $f(x) = \sin \pi x$ (we take radian, not degree).
    - Is f injective?
    - Is f surjective?
    - Is f bijective?
15. Let $X = [1, 10], Y = [0, 1]$ and $f : X \to Y$ defined as $f(x) = \log_{10} x$ (logarithm in base-10).
    - Is f injective?
    - Is f surjective?
    - Is f bijective?
16. True or false. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are bijections, then $f + g$ is also a bijection.

17. Given $f(x) = x^2 - 1$ and $g(x) = x + 1$.
    - Let s be the composition $f(g(x))$. Find the domain of s, the codomain of s, the range of s.
    - Let r be the composition $g(f(x))$. Find the domain of r, the codomain of r, the range of r.
18. Given $f(x) = x + 1$ and $g(x) = \sqrt{x-1}$.
    - Let s be the composition $f(g(x))$. If we limit the codomain of s to $\mathbb{R}$, find the domain of s, the range of s.
    - Let r be the composition $g(f(x))$. If we limit the codomain of r to $\mathbb{R}$, find the domain of r, the range of r.
19. Define inverse function.
20. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = 3x - 2$. Find $f^{-1}(-1)$.
21. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2$. Find $f^{-1}(4)$ and $f^{-1}(9)$
22. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = \dfrac{2}{3}x - 4$. Find f^{-1}, then find $f^{-1} \circ f^{-1}$.
23. Let $a, b, c, d \in \mathbb{R}$ with $a \neq 0$ and $b \neq 0$. Let two functions $f, g : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = ax + b$ and $g(x) = cx + d$. Find $f^{-1} \circ g^{-1}$ and $g^{-1} \circ f^{-1}$.
24. Let $x \in \mathbb{R}$, define $f(x) = \sqrt{x+1}$, $g(x) = x^2 - 1$ and $h(x) = \sqrt{x}$. Find
    - $f + g + h$
    - $f \circ g$
    - $g \circ h$
    - $f \cdot \dfrac{g}{h}$
    - $h \circ f \circ g$
25. Let $x \in \mathbb{R}$, define $f(x) = \dfrac{x-3}{x-1}$. Find $f(f(f(x)))$
26. Let $x \in \mathbb{R}$ and $f(x) = \dfrac{x+1}{x-1}$ and $g(x) = f^{-1}(x)$. Find $g(f(x))$ and $\operatorname{dom} g(f(x))$.
27. Let $x \in [0, 1]$ and $n \in \mathbb{N}_+$. Let $f(x) = (1 - x^n)^{\frac{1}{n}}$. Show that this function is its own inverse function, then find $(f \circ f)(x)$.
28. Let $x \in [0, 1]$. Let $f(x) = (8 - x^3)^{\frac{1}{3}}$. Show that this function is its own inverse function
29. Let $f(x) = x^2$. Let two sets $A = [-1, 1]$ and $B = [-2, 2]$. Find the two sets $f(A)$ and $f(B)$, is $A \subset B$ implies $f(A) \subset f(B)$?
30. Let $f(x) = x^2$. Let two sets $A = [-1, 1]$ and $B = [-2, 2]$. Find the sets $f(A)$, $f(B)$, $f(A \cap B)$ and $f(A) \cap f(B)$. Is $f(A \cap B) \subset f(A) \cap f(B)$?
31. Let $f(x) = x^3$. Let two sets $A = [-1, 1]$ and $B = [-2, 2]$. Find the sets $f(A)$, $f(B)$, $f(A \cup B)$ and $f(A) \cup f(B)$. Is $f(A \cup B) = f(A) \cup f(B)$?
32. Let $f(x) = x^4$. Let two sets $A = [-1, 1]$ and $B = [-2, 2]$. Is $f(A \setminus B) \supset f(A) \setminus f(B)$?

2.2 Function solution

1. Let $\mathcal{A} = \{0, 1, 2, 3\}, \mathcal{B} = \{1, 3, 5, 7\}$.
   - Write the set $\mathcal{A} \times \mathcal{B}$
   $$\mathcal{A} \times \mathcal{B} = \Big\{(0,1),(0,3),(0,5),(0,7)(1,1),(1,3),(1,5),(1,7)(2,1),(2,3),(2,5),(2,7)(3,1),(3,3),(3,5),(3,7)\Big\}$$
   - Let $f : \mathcal{A} \to \mathcal{B}$ given by $f(x) = 2x + 1$. Is f injective? Is f surjective? Is f bijective?
   The set of ordered pairs defined by f is $\Big\{(0,1),(1,3),(2,5),(3,7)\Big\}$, it satisfies the definition of injective, surjective, and it is thus bijective.

2. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Let $f : X \to Y$ as follows: $f(a) = y, f(c) = x$. Is f a function?
   No, nothing is assigned to $b \in X$

3. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Let $f : X \to Y$ as follows: $f(a) = y, f(b) = z, f(c) = x, f(c) = z$. Is f a function?
   No, $c \in X$ is assigned twice

4. Find the domain and range of $f(x) = 9 - x^2$
   The function is well defined in the whole $\mathbb{R}$, hence $\text{dom} f = \mathbb{R}$
   For range, from high school math we know this function can never touch values beyond positive 9, so the range is $\mathbb{R} \setminus (9, +\infty)$, or more compactly, we can write $\{y \mid y \leq 9\}$

5. Find the domain and range of $f(x) = \dfrac{3}{3 - x}$
   For domain, the function is undefined at $x = 3$, hence $\text{dom} f = \mathbb{R} \setminus \{3\}$
   For range, a bit tricky. The function never reach $y = 0$. To see this, if $f(x) = 0$, we have $\dfrac{3}{3 - x} = 0$. No $x \in \mathbb{R}$ gives $f(x) = 0$, thus the range is $\mathbb{R} \setminus \{0\}$. Or equivalently, we also write $(-\infty, 0) \cup (0, +\infty)$

6. Find the domain and range of $f(x) = \dfrac{x - 2}{x + 4}$
   For domain, we look for what values of x will make f undefined.
   For this f, it is undefined if denominator is zero.
   Here we have $x + 4 = 0$ giving $x = -4$, hence $\text{dom} f = \mathbb{R} \setminus \{-4\}$.

   For range, $\mathbb{R} \setminus \{1\}$.
   To find the range, we need to find the values y that is impossible to obtain from all possible x.
   Suppose $y = \dfrac{x - 2}{x + 4}$, then we have $y(x + 4) = x - 2$. Rearrange gives $yx + 4y = x - 2$ and then $(y - 1)x + 4y = -2$. Now if $y = 1$ we get $4 = -2$ which is error.
   In fact, if we put $y = 1$ in $y = \dfrac{x - 2}{x + 4}$, we have $1 = \dfrac{x - 2}{x + 4}$. You can see that no x can gives $\dfrac{x - 2}{x + 4} = 1$, because we will cancel x from $x - 2 = x + 4$

7. Find the domain of $f(x) = \sqrt{\dfrac{1 + x}{1 - x}}$ if the range of f is restricted to $\mathbb{R}$
   There is square root so we have to avoid negative numbers. $\{x \in \mathbb{R} \ : \ -1 \leq x < 1\}$, because if $x = 1$, the denominator is 0 so f is undefined. If $x > 1$, then f output a complex number. If $x < -1$, then f output a complex number.

8. Find the domain and range of $f(x) = 5 + \sqrt{16 - x}$
   Square root cannot have negative values, so we need $16 - x \geq 0$, which give $x \leq 16$.
   Hence $\text{dom} f = \{x \mid x \leq 16\}$

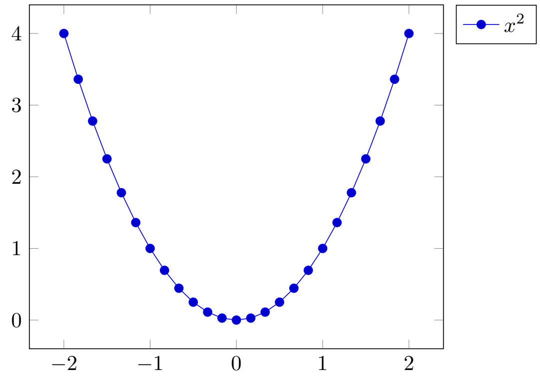
   For range, $\{y \mid y \geq 5\}$

9. Find the domain and range of $f(x) = -\sqrt{81 - x^2}$
   $\text{dom} f = \{x \mid -9 \leq x \leq 9\}$
   Range is $\{y \mid -9 \leq y \leq 0\}$

10. Find the domain and range of $f(x) = x + |x| + 2$
    $\text{dom} f = \mathbb{R}$
    Range is $\{y \mid -y \leq 2\}$

11. Find the domain and range of $f(x) = 3\cos(4x)$
    $\text{dom} f = \mathbb{R}$
    Range is $\{y \mid -3 \leq y \leq 3\}$

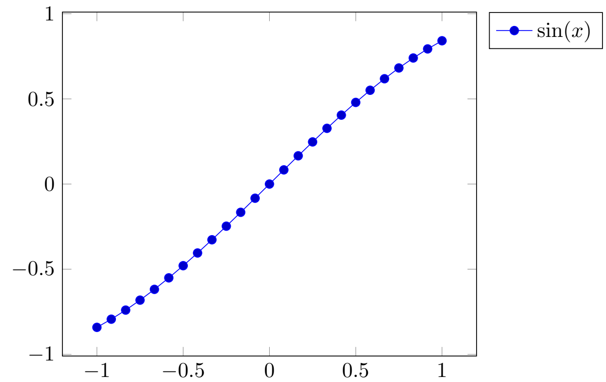
12. Let $X = [-2, 2], Y = [-1, 4]$ and $f : X \to Y$ defined as $f(x) = x^2$.



- Is f injective? yes
- Is f surjective? no
- Is f bijective? no

All $x \in X$ has mapped to a value (injective)
Not all y is mapped (not surjective)
For example, $x^2 \neq -1$, so $-1 \in Y$ is never mapped
not surjective $\implies$ not bijective.

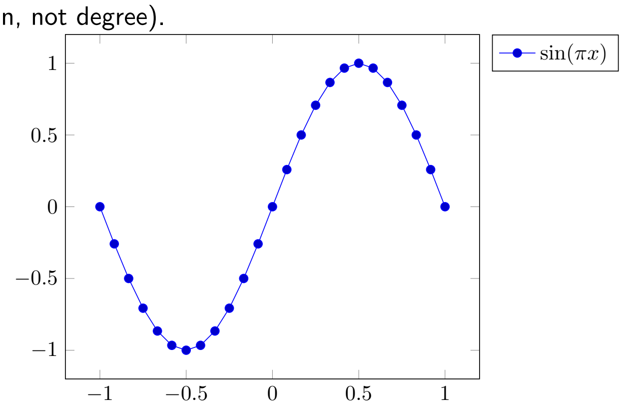
13. Let $X = [-1, 1]$ and $f : X \to X$ defined as $f(x) = \sin x$ (we take radian, not degree).



- Is f injective? yes
- Is f surjective? no
- Is f bijective? no

All $x \in X$ has mapped to a value (injective)
Not all y is mapped (not surjective)
For example, $\sin 1 = 0.84 < 1$, so $1 \in X$ is never mapped
not surjective $\implies$ not bijective.

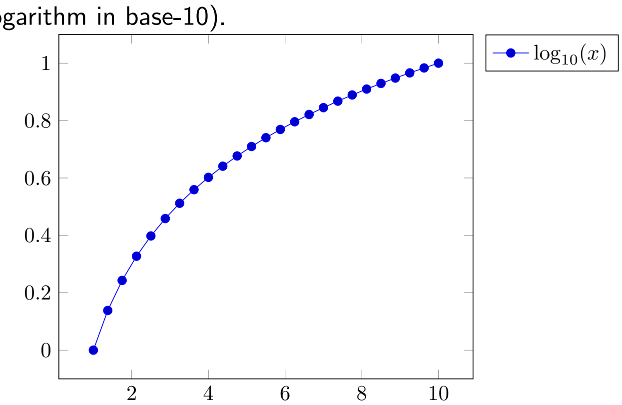
14. Let $X = [-1, 1]$ and $f : X \to X$ defined as $f(x) = \sin \pi x$ (we take radian, not degree).



- Is f injective? yes
- Is f surjective? yes
- Is f bijective? yes

All $x \in X$ has mapped to a value (injective)
All $y \in Y := X$ is mapped (surjective)
injective and surjective $\implies$ bijective.

15. Let $X = [1, 10], Y = [0, 1]$ and $f : X \to Y$ defined as $f(x) = \log_{10} x$ (logarithm in base-10).



- Is f injective? yes
- Is f surjective? yes
- Is f bijective? yes

All $x \in X$ has mapped to a value (injective)
All $y \in Y$ is mapped (surjective)
injective and surjective $\implies$ bijective.

16. True or false. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are bijections, then $f + g$ is also a bijection.
False. To show a statement is false we only need to give a counter example. (Recall the negation of $\forall x P(x)$ is $\exists x \neg P(x)$).
Consider $f(x) = x$ and $g(x) = -x$, both are bijection.
Now $f(x) + g(x) = 0 \forall x$, it is not surjective (only $0 \in \mathbb{R}$ is being mapped) so it is not bijective

17. Given $f(x) = x^2 - 1$ and $g(x) = x + 1$.

- Let s be the composition $f(g(x))$. Find the domain of s, the codomain of s, the range of s.
$$s(x) = f(g(x)) = (g(x))^2 - 1 = (x + 1)^2 - 1 = x^2 - 2x, \qquad \mathrm{dom}\, s = \text{codom}\, s = \mathbb{R}, \qquad \text{range}\, s = [-1, +\infty)$$
- Let r be the composition $g(f(x))$. Find the domain of r, the codomain of r, the range of r.
$$r(x) = g(f(x)) = f(x) + 1 = x^2 \qquad \mathrm{dom}\, r = \text{codom}\, r = \mathbb{R}, \qquad \text{range}\, r = [0, +\infty)$$

18. Given $f(x) = x + 1$ and $g(x) = \sqrt{x-1}$.

 - Let s be the composition $f(g(x))$. If we limit the codomain of s to $\mathbb{R}$, find the domain of s, the range of s.

$$s(x) = f(g(x)) = g(x) + 1 = \sqrt{x-1} + 1$$

 The domain of s is all the real number such that $s(x)$ gives a real but not complex number. Hence $x > 1$, thus $\operatorname{dom} s = [1, +\infty)$.
 The range of s is $[1, +\infty)$.

 - Let r be the composition $g(f(x))$. If we limit the codomain of r to $\mathbb{R}$, find the domain of r, the range of r.

$$r(x) = g(f(x)) = \sqrt{f(x) - 1} = \sqrt{x}$$

 $\operatorname{dom} r = [0, +\infty)$.
 The range of r is $[0, +\infty)$.

19. Define inverse function.
 f is bijective (1-to-1, onto) with domain A and image B. Then the inverse function, denoted by f^{-1}, has domain B and image A, defined by

$$f^{-1}(y) = x \iff f(x) = y \ \ \forall y \in B$$

20. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = 3x - 2$. Find $f^{-1}(-1)$.

$$f^{-1}(x) = \frac{x+2}{3}, \quad f^{-1}(-1) = \frac{1}{3}$$

21. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2$. Find $f^{-1}(4)$ and $f^{-1}(9)$

$$f^{-1}(4) = 2 \qquad f^{-1}(9) = 3$$

22. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = \frac{2}{3}x - 4$. Find f^{-1}, then find $f^{-1} \circ f^{-1}$.

$$f^{-1}(y) = \frac{3}{2}(y+4) \quad \text{or} \quad f^{-1}(x) = \frac{3}{2}(x+4)$$

$$f^{-1} \circ f^{-1} = \frac{3}{2}\Big[\frac{3}{2}(x+4) + 4\Big] = \frac{9}{4}x + 15$$

23. Let $a, b, c, d \in \mathbb{R}$ with $a \neq 0$ and $b \neq 0$. Let two functions $f, g : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = ax + b$ and $g(x) = cx + d$. Find $f^{-1} \circ g^{-1}$ and $g^{-1} \circ f^{-1}$.

$$\begin{aligned} f^{-1} &= \frac{x-b}{a} \\ g^{-1} &= \frac{x-d}{c} \\ f^{-1} \circ g^{-1} &= \frac{x-d-bc}{ac} \\ g^{-1} \circ f^{-1} &= \frac{x-b-ad}{ac} \end{aligned}$$

24. Let $x \in \mathbb{R}$, define $f(x) = \sqrt{x+1}$, $g(x) = x^2 - 1$ and $h(x) = \sqrt{x}$. Find

 - $f + g + h = \sqrt{x+1} + x^2 - 1 + \sqrt{x}$
 - $f \circ g = \sqrt{(x^2-1)+1} = \sqrt{x^2} = |x|$
 - $g \circ h = (\sqrt{x})^2 - 1 = x - 1$
 - $f \cdot \frac{g}{h} = \frac{\sqrt{x+1}(x^2-1)}{\sqrt{x}}$
 - $h \circ f \circ g = \sqrt{|x|}$

25. Let $x \in \mathbb{R}$, define $f(x) = \frac{x-3}{x-1}$. Find $f(f(f(x)))$

$$f(f(x)) = \frac{\frac{x-3}{x-1} - 3}{\frac{x-3}{x-1} - 1} = \frac{x+3}{1-x}$$

$$f(f(f(x))) = \frac{\frac{x+3}{1-x} - 3}{\frac{x+3}{1-x} - 1} = x$$

26. Let $x \in \mathbb{R}$ and $f(x) = \dfrac{x+1}{x-1}$ and $g(x) = f^{-1}(x)$. Find $g(f(x))$ and $\operatorname{dom} g(f(x))$.
First we find the expression of g.

$$y = \frac{x+1}{x-1} \quad \Longrightarrow \quad x = \frac{y+1}{y-1} \quad \Longrightarrow \quad g(x) = f^{-1}(x) = \frac{x+1}{x-1} \text{ if we use the notation on } x$$

Then

$$g(f(x)) \;=\; \frac{f(x)+1}{f(x)-1} \;=\; \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} \;=\; \frac{x+1+x-1}{x+1-x+1} = \frac{x}{2}$$

So $\operatorname{dom} g(f(x)) = \mathbb{R}$

27. Let $x \in [0,1]$ and $n \in \mathbb{N}_+$. Let $f(x) = (1-x^n)^{\frac{1}{n}}$. Show that this function is its own inverse function, then find $(f \circ f)(x)$.

$$y = (1-x^n)^{\frac{1}{n}} \quad \Longrightarrow \quad y^n = (1-x^n) \quad \Longrightarrow \quad x^n = 1 - y^n \quad \Longrightarrow \quad x = (1-y^n)^{\frac{1}{n}}$$

So $f^{-1}(x) = (1-x^n)^{\frac{1}{n}} = f(x)$.
Now we work on $(f \circ f)(x)$

$$(f \circ f)(x) = f(f(x)) = \Big(1 - f(x)^n\Big)^{\frac{1}{n}} = \Big(1 - \big((1-x^n)^{\frac{1}{n}}\big)^n\Big)^{\frac{1}{n}} = x$$

So we have $(f \circ f)(x) = x$.
Note: this is exactly another way to show this function is its own inverse function, because here we have $f \circ f = \text{Id}$ so $f^{-1} = f$

28. Let $x \in [0,1]$. Let $f(x) = (8-x^3)^{\frac{1}{3}}$. Show that this function is its own inverse function
We do so by showing $(f \circ f)(x) = x$

$$(f \circ f)(x) = \sqrt[3]{8 - (\sqrt[3]{8-x^3})^3} = x$$

Hence we have $f \circ f = \text{Id}$ so $f^{-1} = f$

29. Let $f(x) = x^2$. Let two sets $A = [-1,1]$ and $B = [-2,2]$. Find the two sets $f(A)$ and $f(B)$, is $A \subset B$ implies $f(A) \subset f(B)$?

$$f(A) = [0,1], \quad f(B) = [0,4], f(A) \subset f(B)$$

30. Let $f(x) = x^2$. Let two sets $A = [-1,1]$ and $B = [-2,2]$. Find the sets $f(A)$, $f(B)$, $f(A \cap B)$ and $f(A) \cap f(B)$. Is $f(A \cap B) \subset f(A) \cap f(B)$?

$$f(A) = [0,1], \quad f(B) = [0,4], \quad f(A \cap B) = f(A), \quad f(A) \cap f(B) = f(A) \quad \text{ so } f(A \cap B) \subset f(A) \cap f(B) \text{ is true}$$

31. Let $f(x) = x^3$. Let two sets $A = [-1,1]$ and $B = [-2,2]$. Is $f(A \cup B) = f(A) \cup f(B)$?

$$f(A) = [-1,1], \quad f(B) = [-4,4], \quad f(A \cup B) = f(B), \quad f(A) \cup f(B) = f(B) \quad \text{ so } f(A \cup B) = f(A) \cup f(B) \text{ is true}$$

32. Let $f(x) = x^4$. Let two sets $A = [-1,1]$ and $B = [-2,2]$. Is $f(B \setminus A) \supset f(B) \setminus f(A)$?

$$f(A) = [0,1], \quad f(B) = [0,16], \quad f(B \setminus A) = f([-2,-1] \cup [1,2]) = [1,16] \cup [1,16] = [1,16]$$

$$f(B) \setminus f(A) = [0,16] \setminus [0,1] = [1,16] \quad \text{ so } f(B \setminus A) = f(B) \setminus f(A), \text{ equality also implies subset}$$

2.3 Limit computation

1. $\lim_{x\to 1} \frac{x^2-1}{x+1}$
2. $\lim_{x\to 2} \frac{x^2-4}{x-2}$
3. $\lim_{x\to 0} \frac{x-1}{x}$
4. $\lim_{x\to 2} \frac{x^3-8}{x-2}$
5. $\lim_{x\to 3} \frac{x^3-8}{x-2}$
6. $\lim_{x\to -1} \frac{x+1}{x^3+8}$
7. $\lim_{x\to y} \frac{x^n-y^n}{x-y}$
8. $\lim_{y\to x} \frac{x^n-y^n}{x-y}$
9. $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}$
10. $\lim_{x\to +\infty} \frac{x^4+11}{25x^4+2x+12}$
11. $\lim_{x\to -\infty} \frac{3x^2+17x-137}{-x^2+36x+29}$
12. $\lim_{x\to 1} \frac{\frac{1}{3x}-\frac{1}{x+2}}{x^2+2x-3}$
13. $\lim_{x\to \infty} \frac{x^2+1}{x-1}$
14. $\lim_{x\to \infty} \frac{x^2+1}{-9x^4-2x}$
15. $\lim_{x\to 0} \frac{x-\sin x}{x-\tan x}$
16. $\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x}$
17. $\lim_{x\to -\infty} \sqrt{e^{2x}+4e^x}+3-e^x$
18. $\lim_{x\to 0} \frac{\tan x}{x}$
19. $\lim_{x\to 0} \frac{x}{\tan x}$
20. $\lim_{x\to 1} \frac{x-1}{\sin x}$
21. $\lim_{x\to 0} \frac{x}{\cos x}$
22. $\lim_{x\to 1} \frac{\cos(x-1)}{x-1}$
23. $\lim_{x\to 0} \frac{\sin kx}{x}$
24. $\lim_{x\to 0} \frac{\sin 3x}{4x}$
25. $\lim_{x\to 0} \frac{\sin 7x}{\sin 5x}$
26. $\lim_{x\to 0} \frac{x(x-\sin x)}{\tan 2x}$
27. $\lim_{x\to \infty} \left(1+\frac{1}{x}\right)^{\frac{x}{2}}$
28. $\lim_{x\to 0^+} (1+x)^{\frac{1}{x}}$
29. $\lim_{x\to \infty} (2+2x)^{\frac{2}{x}}$
30. $\lim_{x\to \infty} \left(1+\frac{3}{x}\right)^{\frac{x}{2}}$
31. $\lim_{x\to 0} (1+x)^{\frac{1}{x}} \frac{\sin x}{x}$
32. $\lim_{x\to -2} \frac{\sin x+2}{x+2} \ln(2-x)$
33. $\lim_{x\to \infty} \frac{x^3-4x^2+3}{3x^3\cos\left(\frac{1}{x}\right)-1}$
34. $\lim_{x\to 0} \frac{1-\cos x}{x^2}$
35. $\lim_{x\to 0} \frac{x\cos x}{(x^2+1)\sin x}$
36. $\lim_{x\to 1} [\ln(3-x)]^{\sin(x-1)}$
37. $\lim_{x\to \infty} \frac{1-e^x}{1-2e^x}$
38. $\lim_{x\to 0} \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}(1+\cos x)\right)$
39. $\lim_{x\to 1} \frac{1}{x-1}-\frac{3}{x^3-1}$
40. $\lim_{x\to -\infty} x+\sqrt{x^2-2x}$
41. $\lim_{x\to \infty} \sqrt{x^2+x}-x$
42. $\lim_{x\to \infty} \frac{x}{\sqrt{x^2-2x}}$
43. $\lim_{x\to 0} \frac{x\ln(1+\sin x)}{1-\sqrt{\cos x}}$

2.4 Limit computation solution

1. $\lim\limits_{x\to1}\dfrac{x^2-1}{x+1} = \lim\limits_{x\to1}\dfrac{(x-1)(x+1)}{x+1} = \lim\limits_{x\to1} x-1 = 0$

2. $\lim\limits_{x\to2}\dfrac{x^2-4}{x-2} = \lim\limits_{x\to2}\dfrac{(x-2)(x+2)}{x-2} = \lim\limits_{x\to2} x+2 = 4$

3. $\lim\limits_{x\to0}\dfrac{x-1}{x}$ does not exist $\nexists \lim\limits_{x\to0}\dfrac{x-1}{x}$

4. $\lim\limits_{x\to2}\dfrac{x^3-8}{x-2} = \lim\limits_{x\to2}\dfrac{(x-2)(x^2+2x+4)}{x-2} = \lim\limits_{x\to2}(x^2+2x+4) = 12$

5. $\lim\limits_{x\to3}\dfrac{x^3-8}{x-2} = \lim\limits_{x\to3}\dfrac{(x-2)(x^2+2x+4)}{x-2} = \lim\limits_{x\to3}(x^2+2x+4) = 19$

6. $\lim\limits_{x\to-1}\dfrac{x+1}{x^3+8} = \lim\limits_{x\to-1}\dfrac{x+1}{(x+1)(x^2-x+1)} = \lim\limits_{x\to-1}\dfrac{1}{x^2-x+1} = \dfrac{1}{3}$

7. $\lim\limits_{x\to y}\dfrac{x^n-y^n}{x-y} = \lim\limits_{x\to y}\dfrac{(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+y^{n-1})}{x-y} = ny^{n-1}$

8. $\lim\limits_{y\to x}\dfrac{x^n-y^n}{x-y} = \lim\limits_{y\to x}\dfrac{y^n-x^n}{y-x} = nx^{n-1}$ by previous question

9. $\lim\limits_{x\to1}\dfrac{1-\sqrt{x}}{1-x} = \lim\limits_{x\to1}\dfrac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})} = \lim\limits_{x\to1}\dfrac{1}{1+\sqrt{x}} = \dfrac{1}{2}$

10. $\lim\limits_{x\to+\infty}\dfrac{x^4+11}{25x^4+2x+12} = \lim\limits_{x\to+\infty}\dfrac{x^4+11}{25x^4+2x+12}\cdot\dfrac{\frac{1}{x^4}}{\frac{1}{x^4}} = \lim\limits_{x\to+\infty}\dfrac{1+\frac{11}{x^4}}{25+\frac{2}{x^3}+\frac{12}{x^4}} = \dfrac{1}{25}$

11. $\lim\limits_{x\to-\infty}\dfrac{3x^2+17x-137}{-x^2+36x+29} = \lim\limits_{x\to-\infty}\dfrac{x^2}{x^2}\dfrac{3+17x^{-1}-137x^{-1}}{-1+36x^{-1}+29x^{-2}} = \dfrac{3+0-0}{-1+0+0} = -3$

12. $\lim\limits_{x\to1}\dfrac{\frac{1}{3x}-\frac{1}{x+2}}{x^2+2x-3} \overset{\text{LH}}{\to} \lim\limits_{x\to1}\dfrac{\frac{x+2-3x}{3x(x+2)}}{x^2+2x-3} = \lim\limits_{x\to1}\dfrac{-2}{(x+3)(3x)(x+2)} = \dfrac{-1}{18}$

13. $\lim\limits_{x\to\infty}\dfrac{x^2+1}{x-1} \overset{\text{LH}}{\to} \lim\limits_{x\to\infty}\dfrac{2x}{1} = \infty$

14. $\lim\limits_{x\to\infty}\dfrac{x^2+1}{-9x^4-2x} \overset{\text{LH}}{\to} \lim\limits_{x\to\infty}\dfrac{2x}{-36x^3-2} = 0$

15. $\lim\limits_{x\to0}\dfrac{x-\sin x}{x-\tan x} \overset{\text{LH}}{\to} \lim\limits_{x\to0}\dfrac{1-\cos x}{1-\sec^2 x} \overset{\text{LH}}{\to} \lim\limits_{x\to0}\dfrac{\sin x}{-2\sec^2 x\tan x} = \lim\limits_{x\to0}\dfrac{\cos^3 x}{-2} = \dfrac{-1}{2}$

16. $\lim\limits_{x\to0}\dfrac{\sin^2 x}{1-\cos x} = \lim\limits_{x\to0}\dfrac{\sin^2 x}{1-\cos x}\cdot\dfrac{1+\cos x}{1+\cos x} = \lim\limits_{x\to0}\dfrac{\sin^2 x(1+\cos x)}{1-\cos^2 x} = \lim\limits_{x\to0}(1+\cos x) = 2$

17. $\lim\limits_{x\to-\infty}\sqrt{e^{2x}+4e^x}+3-e^x = \sqrt{0+0}+3-0 = 3$

18. $\lim\limits_{x\to0}\dfrac{\tan x}{x} = \lim\limits_{x\to0}\dfrac{\sin}{x}\cdot\dfrac{1}{\cos x} = \lim\limits_{x\to0}\dfrac{\sin}{x}\cdot\lim\limits_{x\to0}\dfrac{1}{\cos x} = 1\cdot1 = 1$

19. $\lim\limits_{x\to0}\dfrac{x}{\tan x} = \lim\limits_{x\to0}\dfrac{1}{\frac{\tan x}{x}} = \dfrac{1}{\lim\limits_{x\to0}\frac{\tan x}{x}} = \dfrac{1}{1} = 1$

20. $\lim\limits_{x\to1}\dfrac{x-1}{\sin x} = \dfrac{1-1}{\sin 1} = 0$

21. $\lim\limits_{x\to0}\dfrac{x}{\cos x} = \dfrac{0}{1} = 0$

22. $\lim\limits_{x\to1}\dfrac{\cos(x-1)}{x-1} \quad \nexists \lim\limits_{x\to1}\dfrac{\cos(x-1)}{x-1}$

23. $\lim\limits_{x\to0}\dfrac{\sin kx}{x} = k\lim\limits_{x\to0}\dfrac{\sin kx}{kx} = k$

24. $\lim\limits_{x\to0}\dfrac{\sin 3x}{4x} = \dfrac{3}{4}\lim\limits_{x\to0}\dfrac{\sin 3x}{3x} = \dfrac{3}{4}$

25. $\lim\limits_{x\to0}\dfrac{\sin7x}{\sin5x}$ $\quad = \lim\limits_{x\to0}\dfrac{\sin7x}{\sin5x}\dfrac{\frac{1}{7x}}{\frac{1}{7x}}\dfrac{\frac{1}{5x}}{\frac{1}{5x}} = \dfrac{7}{5}\lim\limits_{x\to0}\dfrac{\frac{\sin7x}{7x}}{\frac{\sin5x}{5x}} = \dfrac{7}{5}$

26. $\lim\limits_{x\to0}\dfrac{x(x-\sin x)}{\tan2x}$ $\quad = \left(\lim\limits_{x\to0}\dfrac{x}{\sin2x}\right)\left(\lim\limits_{x\to0}\cos2x\right)\left(\lim\limits_{x\to0}(x-\sin x)\right) = \underbrace{\left(\lim\limits_{x\to0}\dfrac{x}{\sin2x}\right)}_{\neq0}\cdot1\cdot0 = 0$

27. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{\frac{x}{2}}$ $\quad = \lim\limits_{x\to\infty}\left(\left(1+\dfrac{1}{x}\right)^{x}\right)^{\frac12} = \left(\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x}\right)^{\frac12} = \sqrt{e}$

28. $\lim\limits_{x\to0^+}(1+x)^{\frac{1}{x}}$ $\quad = \lim\limits_{y\to\infty}\left(1+\dfrac{1}{y}\right)^{y} = e$

29. $\lim\limits_{x\to\infty}(2+2x)^{\frac{2}{x}}$ $\quad = \lim\limits_{x\to\infty}2^{\frac{2}{x}}(1+x)^{\frac{2}{x}} = \left(\lim\limits_{x\to\infty}2^{\frac{2}{x}}\right)\left(\lim\limits_{x\to\infty}(1+x)^{\frac{1}{x}}\right)^2 = 1\cdot e^2 = e^2$

30. $\lim\limits_{x\to\infty}\left(1+\dfrac{3}{x}\right)^{\frac{x}{2}}$ $\quad = \lim\limits_{y\to\infty}\left(1+\dfrac{1}{y}\right)^{3y/2} = e^{3/2}$

31. $\lim\limits_{x\to0}(1+x)^{\frac{1}{x}}\dfrac{\sin x}{x}$ $\quad = \left(\lim\limits_{x\to0}(1+x)^{\frac{1}{x}}\right)\left(\lim\limits_{x\to0}\dfrac{\sin x}{x}\right) = e$

32. $\lim\limits_{x\to-2}\dfrac{\sin x+2}{x+2}\ln(2-x)$ $\quad = \left(\lim\limits_{x\to-2}\dfrac{\sin x+2}{x+2}\right)\left(\lim\limits_{x\to-2}\ln(2-x)\right) = \left(\lim\limits_{t\to0}\dfrac{\sin t}{t}\right)\ln(4) = \ln4$

33. $\lim\limits_{x\to\infty}\dfrac{x^3-4x^2+3}{3x^3\cos\left(\frac{1}{x}\right)-1}$ $\quad = \lim\limits_{x\to\infty}\dfrac{x^3-4x^2+3}{3x^3\cos\left(\frac{1}{x}\right)-1} = \lim\limits_{x\to\infty}\dfrac{x^3+0}{3x^3-0} = \dfrac{1}{3}$

34. $\lim\limits_{x\to0}\dfrac{1-\cos x}{x^2}$ $\quad = \lim\limits_{x\to0}\dfrac{1-\cos x}{x^2}\dfrac{1+\cos x}{1+\cos x} = \lim\limits_{x\to0}\dfrac{\sin^2x}{x^2}\dfrac{1}{2\cos^2\left(\frac{x}{2}\right)} = \dfrac{1}{2}$

35. $\lim\limits_{x\to0}\dfrac{x\cos x}{(x^2+1)\sin x}$ $\quad = \lim\limits_{x\to0}\dfrac{x}{\sin x}\lim\limits_{x\to0}\dfrac{\cos x}{x^2+1} = 1\cdot\dfrac{1}{0+1} = 1$

36. $\lim\limits_{x\to1}[\ln(3-x)]^{\sin(x-1)}$ $\quad = \lim\limits_{x\to1}[\ln(3-1)]^0 = 1$

37. $\lim\limits_{x\to\infty}\dfrac{1-e^x}{1-2e^x}$ $\quad \lim\limits_{x\to\infty}\dfrac{\dfrac{1}{e^x}-1}{\dfrac{1}{e^x}-2} = \dfrac{0-1}{0-2} = \dfrac{1}{2}$

38. $\lim\limits_{x\to0}\left(\dfrac{1}{x}\dfrac{\mathrm{d}}{\mathrm{d}x}(1+\cos x)\right)$ $\quad \lim\limits_{x\to0}\left(\dfrac{-\sin x}{x}\right) = -1$

39. $\lim\limits_{x\to1}\dfrac{1}{x-1}-\dfrac{3}{x^3-1}$

$$\lim_{x\to1}\frac{1}{x-1}-\frac{3}{x^3-1}\quad\to\infty-\infty$$
$$=\lim_{x\to1}\frac{1}{x-1}\left(1-\frac{3}{x^2+x+1}\right)$$
$$=\lim_{x\to1}\frac{1}{x-1}\left(\frac{x^2+x+1-3}{x^2+x+1}\right) = \lim_{x\to1}\frac{1}{x-1}\left(\frac{x^2+x-2}{x^2+x+1}\right) = \lim_{x\to1}\frac{1}{x-1}\left(\frac{(x-1)(x+2)}{x^2+x+1}\right) = \lim_{x\to1}\frac{x+2}{x^2+x+1} = 1$$

40. $\lim\limits_{x\to-\infty}x+\sqrt{x^2-2x}$

$$=\lim_{x\to-\infty}\frac{(x+\sqrt{x^2-2x})(x-\sqrt{x^2-2x})}{x-\sqrt{x^2-2x}} = \lim_{x\to-\infty}\frac{2x}{x-\sqrt{x^2-2x}} = \lim_{x\to-\infty}\frac{x}{1-\sqrt{1-2/x}} = 1$$

41. $\lim\limits_{x\to\infty}\sqrt{x^2+x}-x$

$$=\lim_{x\to\infty}\frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x} = \lim_{x\to\infty}\frac{x^2+x-x^2}{\sqrt{x^2+x}+x} = \lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x} = \lim_{x\to\infty}\frac{1}{\sqrt{1+1/x}+1} = \frac{1}{2}$$

42. $\lim\limits_{x\to\infty}\dfrac{x}{\sqrt{x^2-2x}}$ $\quad = \lim\limits_{x\to\infty}\dfrac{x}{\sqrt{x^2-2x}}\cdot\dfrac{1/x}{1/x} = \lim\limits_{x\to\infty}\dfrac{1}{\sqrt{\frac{x^2-2x}{x^2}}} = \lim\limits_{x\to\infty}\dfrac{1}{\sqrt{1-2/x}} =$

43. $\lim\limits_{x\to 0} \dfrac{x \ln(1+\sin x)}{1-\sqrt{\cos x}}$

$$\begin{aligned}
&= \lim_{x\to 0} \frac{x\ln(1+\sin x)}{1-\sqrt{\cos x}} \cdot \frac{1+\sqrt{\cos x}}{1+\sqrt{\cos x}} && \text{multiply conjugate of } 1-\sqrt{\cos x} \\
&= \lim_{x\to 0} \frac{x\ln(1+\sin x)(1+\sqrt{\cos x})}{1-\cos x} \\
&= \lim_{x\to 0} \frac{x\ln(1+\sin x)(1+\sqrt{\cos x})}{1-\cos x} \cdot \frac{1+\cos x}{1+\cos x} && \text{multiply conjugate of } 1-\cos x \\
&= \lim_{x\to 0} \frac{x\ln(1+\sin x)(1+\sqrt{\cos x})(1+\cos x)}{1-\cos^2 x} \\
&= \lim_{x\to 0} \frac{x(1+\sqrt{\cos x})(1+\cos x)\ln(1+\sin x)}{\sin^2 x} \\
&= \lim_{x\to 0} \frac{x}{\sin x}(1+\sqrt{\cos x})(1+\cos x)\frac{\ln(1+\sin x)}{\sin x} \\
&= \left(\lim_{x\to 0} \frac{x}{\sin x}\right)\left(\lim_{x\to 0}(1+\sqrt{\cos x})(1+\cos x)\right)\left(\lim_{x\to 0} \frac{\ln(1+\sin x)}{\sin x}\right) && \text{all the bracket has limit so we can distribute} \\
&= \left(\lim_{x\to 0} \frac{1}{\frac{\sin x}{x}}\right)\left((1+\sqrt{1})(1+1)\right)\left(\lim_{t\to 0} \frac{\ln(1+t)}{t}\right) && x\to 0 \iff \sin x \to 0 \\
&= \frac{4}{\lim\limits_{x\to 0} \dfrac{\sin x}{x}} \\
&= \frac{4}{1} = 4
\end{aligned}$$

2.5 L'Hopital's rule

1. $\lim\limits_{x\to 0} \dfrac{x^2}{\ln \sec x}$
2. $\lim\limits_{x\to -1} \dfrac{x+1}{x^3+1}$
3. $\lim\limits_{x\to 0} \dfrac{\ln(1+x^3)}{x-\sin(x)}$
4. $\lim\limits_{x\to \infty} \dfrac{x^3}{1^2+2^2+3^2+\cdots+x^2}$
5. $\lim\limits_{t\to 0} \dfrac{e^{2t}-2t-1}{-\cos(3t)+1}$
6. $\lim\limits_{x\to 0} \dfrac{\ln(1-x)+x+x^2/2}{x^3}$
7. $\lim\limits_{x\to \infty} \sqrt{e^{2x}+4e^x}+3-e^x$
8. $\lim\limits_{x\to \infty} x\Big(1-\cos\big(\dfrac{1}{x}\big)\Big)$
9. $\lim\limits_{t\to 0} \dfrac{t^2+2\ln(\cos t)}{t^4}$
10. $\lim\limits_{x\to 0} \dfrac{\tan x-\sin x}{x^3}$
11. $\lim\limits_{t\to 0} \dfrac{\dfrac{1}{\sqrt{9+2t}}-\dfrac{1}{3}}{\sin 3t}$
12. $\lim\limits_{x\to 5} \dfrac{\sqrt{x-1}-\sqrt{\sqrt{2x+6}}}{\sin(x-5)}$
13. $\lim\limits_{x\to \infty} \sin\Big(\dfrac{2}{x}\Big)\Big(\sqrt{x^4+5x^3}-x^2\Big)$
14. $\lim\limits_{x\to \infty} \dfrac{4e^x+e^{2x}\sin(3e^{-x})}{3e^x+7e^{-x}+5}$
15. $\lim\limits_{x\to 0} \Big[e^2\ln(2+x)^{\frac{1}{x}}\sin x\Big]$
16. $\lim\limits_{x\to \infty} \Big(\dfrac{x+1}{x-1}\Big)^x$
17. $(**)$ $\lim\limits_{x\to 1} \dfrac{\sin(2x-2)}{x^2-1}$ (try also without using L'Hosptial rule)
18. $\lim\limits_{x\to 0} \left(2024+\dfrac{2023}{x}\right)^{2024x}$
19. $\lim\limits_{x\to \infty} \dfrac{3e^{2x}+e^x-x^4}{4e^{2x}-5e^x+2x^4}$
20. $(\star\star\star)$ $\lim\limits_{t\to 3} \dfrac{\sin\big(\sin(t-3)\big)}{2t+\sqrt{5t+1}-10}$
21. $(****)$ $\lim\limits_{t\to 0^+} \dfrac{5t^2\sin^2(\ln t^2)+4t}{t-\sqrt{t+4}+2}$
22. $\lim\limits_{x\to \infty} x\ln\Big(\dfrac{x+1}{x-1}\Big)$
23. $\lim\limits_{x\to 0^+} x\ln\big(\sin x\big)$
24. $\lim\limits_{x\to 0^+} x^{\sin x}$

2.6 L'Hopital's rule solution

1. $\displaystyle\lim_{x\to 0}\frac{x^2}{\ln\sec x}$

$$\begin{aligned}
&\lim_{x\to 0}\frac{x^2}{\ln\sec x} && \to \frac{0}{0}\\
&\overset{L.H.}{=} \lim_{x\to 0}\frac{2x}{\frac{\sec x\tan x}{\sec x}}\\
&\overset{L.H.}{=} \lim_{x\to 0}\frac{2x}{\tan x} && \to \frac{0}{0}\\
&\overset{L.H.}{=} \lim_{x\to 0}\frac{2}{\sec^2 x}\\
&\overset{L.H.}{=} 2\lim_{x\to 0}\cos^2 x\\
&= 2
\end{aligned}$$

2. $\displaystyle\lim_{x\to -1}\frac{x+1}{x^3+1}$

$$\begin{aligned}
&\lim_{x\to -1}\frac{x+1}{x^3+1} && \to \frac{0}{0}\\
&\overset{L.H.}{=} \lim_{x\to -1}\frac{1}{3x^2} = \frac{1}{3}
\end{aligned}$$

3. $\displaystyle\lim_{x\to 0}\frac{\ln(1+x^3)}{x-\sin(x)}$

$$\begin{aligned}
&\lim_{x\to 0}\frac{\ln(1+x^3)}{x-\sin(x)} && \to \frac{0}{0}\\
&\overset{L.H.}{=} \lim_{x\to 0}\frac{\frac{x^2}{1+x^3}}{1-\cos x}\\
&= \lim_{x\to 0}\frac{1}{1+x^3}\lim_{x\to 0}\frac{x^2}{1-\cos x}\\
&= \lim_{x\to 0}\frac{x^2}{1-\cos x} && \to \frac{0}{0}\\
&\overset{L.H.}{=} \lim_{x\to 0}\frac{2x}{\sin x} = 2
\end{aligned}$$

4. $\displaystyle\lim_{x\to\infty}\frac{x^3}{1^2+2^2+3^2+\cdots+x^2}$

$$\begin{aligned}
&\lim_{x\to\infty}\frac{x^3}{1^2+2^2+3^2+\cdots+x^2} && \to \frac{\infty}{\infty}\\
&= \lim_{x\to\infty}\frac{x^3}{\frac{1}{3}\left(x^3+\frac{3}{2}x^2+\frac{3}{6}x\right)} && \text{using formula of sum of quadratics (not trivial)}\\
&= 3
\end{aligned}$$

5. $\displaystyle\lim_{t\to 0}\frac{e^{2t}-2t-1}{-\cos(3t)+1}$

$$\begin{aligned}
&\lim_{t\to 0}\frac{e^{2t}-2t-1}{-\cos(3t)+1} && \to \frac{e^0-0-1}{1-1}=\frac{0}{0}\\
&\overset{L.H.}{=} \lim_{t\to 0}\frac{2e^{2t}-2}{3\sin(3t)} && \to \frac{2-2}{0}=\frac{0}{0}\\
&\overset{L.H.}{=} \lim_{t\to 0}\frac{4e^{2t}}{9\cos(3t)} = \frac{4}{9}
\end{aligned}$$

6. $\lim_{x\to 0} \frac{\ln(1-x)+x+x^2/2}{x^3}$

$$\begin{aligned}
&\lim_{x\to 0} \frac{\ln(1-x)+x+x^2/2}{x^3} && \to \frac{0}{0}\\
&\overset{\text{LH}}{\to} \lim_{x\to 0} \frac{\frac{-1}{1-x}+1+x}{3x^2} && \to \frac{0}{0}\\
&\overset{\text{LH}}{\to} \lim_{x\to 0} \frac{\frac{-1}{(1-x)^2}+1}{6x} && \to \frac{0}{0}\\
&\overset{\text{LH}}{\to} \lim_{x\to 0} \frac{\frac{-2}{(1-x)^3}}{6} = \frac{-1}{3}
\end{aligned}$$

7. $\lim_{x\to\infty} \sqrt{e^{2x}+4e^x}+3-e^x$

$$\begin{aligned}
&\lim_{x\to\infty} \sqrt{e^{2x}+4e^x}+3-e^x && \to \infty-\infty\\
&= \lim_{x\to\infty} \sqrt{e^{2x}+4e^x}+(3-e^x)\cdot \frac{\sqrt{e^{2x}+4e^x}-(3-e^x)}{\sqrt{e^{2x}+4e^x}-(3-e^x)}\\
&= \lim_{x\to\infty} \frac{e^{2x}+4e^x-(3-e^x)^2}{\sqrt{e^{2x}+4e^x}-(3-e^x)}\\
&= \lim_{x\to\infty} \frac{e^{2x}+4e^x-(9-6e^x+e^{2x})}{\sqrt{e^{2x}+4e^x}-(3-e^x)}\\
&= \lim_{x\to\infty} \frac{10e^x-9}{\sqrt{e^{2x}(1+4e^{-x})}-(3-e^x)}\\
&= \lim_{x\to\infty} \frac{10e^x-9}{e^x\sqrt{1+4e^{-x}}-(3-e^x)}\cdot\frac{1/e^x}{1/e^x}\\
&= \lim_{x\to\infty} \frac{10-\frac{9}{e^x}}{\sqrt{1+4e^{-x}}-\frac{3}{e^x}+1}\\
&= \frac{10}{\sqrt{1}+1} = 5
\end{aligned}$$

8. $\lim_{x\to\infty} x\Big(1-\cos\big(\frac{1}{x}\big)\Big)$

$$\begin{aligned}
&\lim_{x\to\infty} x\Big(1-\cos\big(\frac{1}{x}\big)\Big) && \to \infty\cdot 0\\
&= \lim_{t\to 0} \frac{1-\cos t}{t} && \text{change of variable } t=\frac{1}{x}\\
&\overset{L.H.}{=} \lim_{t\to 0} \frac{\sin t}{1} = 0
\end{aligned}$$

9. $\lim_{t\to 0} \frac{t^2+2\ln(\cos t)}{t^4}$

$$\begin{aligned}
&\lim_{t\to 0} \frac{t^2+2\ln(\cos t)}{t^4} && \to \frac{0}{0}\\
&\overset{L.H.}{=} \lim_{t\to 0} \frac{2t+2\frac{-\sin t}{\cos t}}{4t^3} && \to \frac{0}{0}\\
&\overset{L.H.}{=} \lim_{t\to 0} \frac{2-2\sec^2 t}{12t^2} && \to \frac{0}{0}\\
&\overset{L.H.}{=} \lim_{t\to 0} \frac{-4\sec t\sec t\tan t}{24t} = \lim_{t\to 0} \frac{-4\sec^2 t\tan t}{24t} && \to \frac{0}{0}\\
&= \lim_{t\to 0} \frac{-8\sec t\sec t\tan t\tan t-4\sec^2 t\sec^2 t}{24} = \frac{-1}{6}
\end{aligned}$$

10. $\lim_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$

$$\begin{aligned}
&\lim_{x\to 0}\frac{\tan x-\sin x}{x^3} && \to \frac{0}{0}\\
&\overset{L.H.}{=}\lim_{x\to 0}\frac{\sec^2 x-\cos x}{3x^2} && \to \frac{0}{0}\\
&\overset{L.H.}{=}\lim_{x\to 0}\frac{2\cos^3 x\sin x+\sin x}{6x} && \to \frac{0}{0}\\
&=\lim_{x\to 0}\frac{\sin x}{x}\frac{2\cos^3 x+1}{6}\\
&=\Big(\lim_{x\to 0}\frac{\sin x}{x}\Big)\Big(\lim_{x\to 0}\frac{2\cos^3 x+1}{6}\Big)\\
&=\frac{1}{2}
\end{aligned}$$

11. $\lim_{t\to 0}\dfrac{\dfrac{1}{\sqrt{9+2t}}-\dfrac{1}{3}}{\sin 3t}$

$$\begin{aligned}
\lim_{t\to 0}\frac{\dfrac{1}{\sqrt{9+2t}}-\dfrac{1}{3}}{\sin 3t} &= \lim_{t\to 0}\frac{\dfrac{3-\sqrt{9+2t}}{3\sqrt{9+2t}}}{\sin 3t} = \lim_{t\to 0}\frac{3-\sqrt{9+2t}}{3\sqrt{9+2t}\sin 3t} = \lim_{t\to 0}\frac{3-\sqrt{9+2t}}{3\sqrt{9+2t}\sin 3t}\cdot\frac{3+\sqrt{9+2t}}{3+\sqrt{9+2t}}\\
&= \lim_{t\to 0}\frac{9-9-2t}{3\sqrt{9+2t}(\sin 3t)(3+\sqrt{9+2t})} = \frac{-2}{3}\lim_{t\to 0}\frac{t}{\sqrt{9+2t}(\sin 3t)(3+\sqrt{9+2t})}\\
&= \frac{-2}{9}\lim_{t\to 0}\frac{3t}{\sin 3t}\cdot\frac{1}{\sqrt{9+2t}(3+\sqrt{9+2t})}\\
&= \frac{-2}{9}\left(\lim_{t\to 0}\frac{1}{\frac{\sin 3t}{3t}}\right)\left(\lim_{t\to 0}\frac{1}{\sqrt{9+2t}(3+\sqrt{9+2t})}\right) = \frac{-2}{9}\left(\frac{1}{\lim_{t\to 0}\dfrac{\sin 3t}{3t}}\right)\left(\frac{1}{3(6)}\right) = \frac{-1}{81}
\end{aligned}$$

12. $\lim_{x\to 5}\dfrac{\sqrt{x-1}-\sqrt{\sqrt{2x+6}}}{\sin(x-5)}$

$$\begin{aligned}
\lim_{x\to 5}\frac{\sqrt{x-1}-\sqrt{\sqrt{2x+6}}}{\sin(x-5)} &= \lim_{x\to 5}\frac{\sqrt{x-1}-\sqrt{\sqrt{2x+6}}}{\sin(x-5)}\cdot\frac{\sqrt{x-1}+\sqrt{\sqrt{2x+6}}}{\sqrt{x-1}+\sqrt{\sqrt{2x+6}}}\\
&= \lim_{x\to 5}\frac{(x-1)-\sqrt{2x+6}}{\sin(x-5)\Big(\sqrt{x-1}+\sqrt{\sqrt{2x+6}}\Big)}\\
&= \lim_{x\to 5}\frac{(x-1)-\sqrt{2x+6}}{\sin(x-5)\Big(\sqrt{x-1}+\sqrt{\sqrt{2x+6}}\Big)}\cdot\frac{(x-1)+\sqrt{2x+6}}{(x-1)+\sqrt{2x+6}}\\
&= \lim_{x\to 5}\frac{(x-1)^2-(2x+6)}{\sin(x-5)\Big(\sqrt{x-1}+\sqrt{\sqrt{2x+6}}\Big)\Big(x-1+\sqrt{2x+6}\Big)}\\
&= \lim_{x\to 5}\frac{x^2-4x-5}{\sin(x-5)\Big(\sqrt{x-1}+\sqrt{\sqrt{2x+6}}\Big)\Big(x-1+\sqrt{2x+6}\Big)}\\
&= \lim_{x\to 5}\frac{(x-5)(x+1)}{\sin(x-5)\Big(\sqrt{x-1}+\sqrt{\sqrt{2x+6}}\Big)\Big(x-1+\sqrt{2x+6}\Big)}\\
&= \lim_{x\to 5}\frac{x-5}{\sin(x-5)}\cdot\frac{x+1}{\Big(\sqrt{x-1}+\sqrt{\sqrt{2x+6}}\Big)\Big(x-1+\sqrt{2x+6}\Big)}\\
&= \left(\lim_{x\to 5}\frac{x-5}{\sin(x-5)}\right)\left(\lim_{x\to 5}\frac{x+1}{\Big(\sqrt{x-1}+\sqrt{\sqrt{2x+6}}\Big)\Big(x-1+\sqrt{2x+6}\Big)}\right)\\
&= 1\cdot\frac{6}{\Big(\sqrt{4}+\sqrt{\sqrt{16}}\Big)\Big(4+\sqrt{16}\Big)} = \frac{6}{4\cdot 8} = \frac{3}{16}
\end{aligned}$$

13. $\lim\limits_{x\to\infty} \sin\left(\frac{2}{x}\right)\left(\sqrt{x^4+5x^3}-x^2\right)$

$$
\begin{aligned}
& \lim_{x\to\infty} \sin\left(\frac{2}{x}\right)\left(\sqrt{x^4+5x^3}-x^2\right) && \to 0\cdot(\infty-\infty) \\
= \; & \lim_{x\to\infty} \sin\left(\frac{2}{x}\right)\left(\sqrt{x^4+5x^3}-x^2\right)\cdot\frac{\sqrt{x^4+5x^3}+x^2}{\sqrt{x^4+5x^3}+x^2} && \text{to remove square root} \\
= \; & \lim_{x\to\infty} \sin\left(\frac{2}{x}\right)\frac{x^4+5x^3-x^4}{\sqrt{x^4+5x^3}+x^2} \\
= \; & \lim_{x\to\infty} \sin\left(\frac{2}{x}\right)\frac{5x^3}{\sqrt{x^4(1+5/x)}+x^2} \\
= \; & \lim_{x\to\infty} \sin\left(\frac{2}{x}\right)\frac{5x^3}{x^2\sqrt{1+5/x}+x^2}\cdot\frac{1/x^2}{1/x^2} \\
= \; & \lim_{x\to\infty} \sin\left(\frac{2}{x}\right)\frac{5x}{\sqrt{1+5/x}+1} \\
= \; & \left(\lim_{x\to\infty}\frac{5}{\sqrt{1+5/x}+1}\right)\left(\lim_{x\to\infty}\sin\left(\frac{2}{x}\right)x\right) \\
= \; & \frac{5}{2}\left(\lim_{x\to\infty}\sin\left(\frac{2}{x}\right)x\cdot\frac{2/x}{2/x}\right) \\
= \; & \frac{10}{2}\lim_{x\to\infty}\frac{\sin\left(\frac{2}{x}\right)}{\frac{2}{x}} \\
= \; & \frac{10}{2}\underbrace{\lim_{u\to 0}\frac{\sin u}{u}}_{=1} \; = \; 5 && \text{substitute } \frac{2}{x}=u
\end{aligned}
$$

14. $\lim\limits_{x\to\infty}\frac{4e^x+e^{2x}\sin(3e^{-x})}{3e^x+7e^{-x}+5}$

$$
\begin{aligned}
\lim_{x\to\infty}\frac{4e^x+e^{2x}\sin(3e^{-x})}{3e^x+7e^{-x}+5} \; &= \; \lim_{x\to\infty}\frac{4e^x+e^{2x}\sin(3e^{-x})}{3e^x+7e^{-x}+5}\cdot\frac{1/e^x}{1/e^x} \\
&= \; \lim_{x\to\infty}\frac{4+e^x\sin(3e^{-x})}{3+7e^{-2x}+5e^{-x}} \\
&= \; \frac{4+\lim\limits_{x\to\infty}e^x\sin(3e^{-x})}{3+\lim\limits_{x\to\infty}7e^{-2x}+5e^{-x}} \\
&= \; \frac{4+\lim\limits_{x\to\infty}e^x\sin(3e^{-x})}{3} \\
&= \; \frac{4+\lim\limits_{x\to\infty}\frac{\sin(3e^{-x})}{e^{-x}}}{3} \\
&= \; \frac{4+3\lim\limits_{x\to\infty}\frac{\sin(3e^{-x})}{3e^{-x}}}{3} \\
&= \; \frac{4+3\lim\limits_{u\to 0}\frac{\sin u}{u}}{3} \\
&= \; \frac{7}{3}
\end{aligned}
$$

15. $\lim\limits_{x\to 0}\left[e^2\ln(2+x)^{\frac{1}{x}}\sin x\right]$

$$
\begin{aligned}
\lim_{x\to 0}\left[e^2\ln(2+x)^{\frac{1}{x}}\sin x\right] \; &= \; e^2\lim_{x\to 0}\left[\frac{1}{x}\ln(2+x)\sin x\right] \\
&= \; e^2\lim_{x\to 0}\left[\ln(2+x)\frac{\sin x}{x}\right] \\
&= \; e^2\Big[\lim_{x\to 0}\ln(2+x)\Big]\Big[\lim_{x\to 0}\frac{\sin x}{x}\Big] \\
&= \; e^2\ln 2
\end{aligned}
$$

16. $\lim\limits_{x\to\infty}\left(\dfrac{x+1}{x-1}\right)^x$

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x &= \lim_{x\to\infty}\left(\frac{x+1}{x-1}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}\right)^x \\
&= \lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)^x}{\left(1-\frac{1}{x}\right)^x} \\
&= \frac{\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x}{\lim\limits_{x\to\infty}\left(1-\dfrac{1}{x}\right)^x} && \text{limit of quotient = quotient of limit if the denominator } \neq 0 \\
&= \frac{e}{\lim\limits_{x\to\infty}\left(\left(1-\dfrac{1}{x}\right)^{-x}\right)^{-1}} \\
&= \frac{e}{\left(\lim\limits_{x\to\infty}\left(1-\dfrac{1}{x}\right)^{-x}\right)^{-1}} \\
&= \frac{e}{\left(\lim\limits_{y\to\infty}\left(1+\dfrac{1}{y}\right)^{y}\right)^{-1}} && \text{let } y=-x \\
&= \frac{e}{e^{-1}} \;=\; e^2
\end{aligned}$$

17. $(**)\ \lim\limits_{x\to 1}\dfrac{\sin(2x-2)}{x^2-1}$

with L'Hosptial's rule

$$\begin{aligned}
\lim_{x\to 1}\frac{\sin(2x-2)}{x^2-1} &= \lim_{x\to 1}\frac{2\cos(2x-2)}{2x} \\
&= \lim_{x\to 1}\frac{\cos(2x-2)}{x} \;=\; \frac{\cos(0)}{1} \;=\; 1
\end{aligned}$$

without using L'Hosptial's rule

$$\begin{aligned}
\lim_{x\to 1}\frac{\sin(2x-2)}{x^2-1} &= \lim_{t\to 0}\frac{\sin(2t)}{t(t+2)} && \text{let } t=x-1 \\
&= \lim_{t\to 0}\left(\frac{\sin(2t)}{2t}-\frac{\sin(2t)}{2(t+2)}\right) && \text{partial fraction} \\
&= \left(\lim_{t\to 0}\frac{\sin(2t)}{2t}\right)-\left(\lim_{t\to 0}\frac{\sin(2t)}{2(t+2)}\right) \\
&= (1)-\left(\frac{\sin(0)}{4}\right) \;=\; 1
\end{aligned}$$

18. $\lim\limits_{x\to 0}\left(2024+\dfrac{2023}{x}\right)^{2024x}$

$$\begin{aligned}
\lim_{x\to 0}\left(2024+\frac{2023}{x}\right)^{2024x} &= \lim_{x\to 0}2024^{2024x}\left(1+\frac{2023}{2024x}\right)^{2024x} \\
&= \left(\lim_{x\to 0}2024^{2024x}\right)\cdot\lim_{x\to 0}\left(1+\frac{2023}{2024x}\right)^{2024x} \\
&= 1\cdot\lim_{x\to 0}\left(1+\frac{2023}{2024x}\right)^{2024x} \\
&= \lim_{t\to 0}\left(1+\frac{1}{t}\right)^{2023t} && t=\frac{2024}{2023}x \\
&= \left(\lim_{t\to 0}\left(1+\frac{1}{t}\right)^{t}\right)^{2023} \;=\; e^{2023}
\end{aligned}$$

19. $\lim\limits_{x\to\infty}\dfrac{3e^{2x}+e^x-x^4}{4e^{2x}-5e^x+2x^4}$

$$\lim_{x\to\infty}\frac{3e^{2x}+e^x-x^4}{4e^{2x}-5e^x+2x^4}=\lim_{x\to\infty}\frac{3+e^{-x}-x^4e^{-2x}}{4-5e^{-x}+2x^4e^{-2x}}\overset{LH}{=}\lim_{x\to\infty}\frac{3+0-0}{4-0+0}=\frac{3}{4}$$

20. $(\star\star\star)\ \lim\limits_{t\to 3} \dfrac{\sin\big(\sin(t-3)\big)}{2t+\sqrt{5t+1}-10}$

$$\begin{aligned}
\lim_{t\to 3} \frac{\sin\big(\sin(t-3)\big)}{\sqrt{5t+1}-(10-2t)} &= \lim_{t\to 3} \frac{\sin\big(\sin(t-3)\big)}{\sqrt{5t+1}-(10-2t)} \cdot \frac{\sqrt{5t+1}+(10-2t)}{\sqrt{5t+1}+(10-2t)} = \lim_{t\to 3} \frac{\sin\big(\sin(t-3)\big)\Big(\sqrt{5t+1}+10-2t\Big)}{5t+1-(10-2t)^2} \\
&= \lim_{t\to 3} \frac{\sin\big(\sin(t-3)\big)\Big(\sqrt{5t+1}+10-2t\Big)}{-4t^2+45x-99} = \lim_{t\to 3} \frac{\sin\big(\sin(t-3)\big)\Big(\sqrt{5t+1}+10-2t\Big)}{-(4t^2-45t+99)} \\
&= \frac{\sin\big(\sin(t-3)\big)\Big(\sqrt{5t+1}+10-2t\Big)}{-(t-3)(4t-33)} \\
&= \lim_{t\to 3} \frac{\sin\big(\sin(t-3)\big)\cdot \sin(t-3) \cdot \Big(\sqrt{5t+1}+10-2t\Big)}{-\sin(t-3)\cdot(t-3)(4t-33)} \\
&= \lim_{t\to 3} \frac{\sin\big(\sin(t-3)\big)}{\sin(t-3)} \cdot \frac{\sin(t-3)}{t-3} \cdot \frac{\sqrt{5t+1}+10-2t}{-(4t-33)} \\
&= \underbrace{\left(\lim_{t\to 3} \frac{\sin\big(\sin(t-3)\big)}{\sin(t-3)}\right)}_{=1} \underbrace{\left(\lim_{t\to 3} \frac{\sin(t-3)}{t-3}\right)}_{=1} \underbrace{\left(\lim_{t\to 3} \frac{\sqrt{5t+1}+10-2t}{-(4t-33)}\right)}_{=8/21} = \frac{8}{21}
\end{aligned}$$

21. $(****)\ \lim\limits_{t\to 0^+} \dfrac{5t^2\sin^2(\ln t^2)+4t}{t-\sqrt{t+4}+2}$

$$\begin{aligned}
\lim_{t\to 0^+} \frac{5t^2\sin^2(\ln t^2)+4t}{t-\sqrt{t+4}+2} &= \lim_{t\to 0^+} \frac{5t^2\sin^2(\ln t^2)+4t}{(t+2)-\sqrt{t+4}} \cdot \frac{(t+2)+\sqrt{t+4}}{(t+2)+\sqrt{t+4}} \\
&= \lim_{t\to 0^+} \frac{\Big(5t^2\sin^2(\ln t^2)+4t\Big)\Big(t+2+\sqrt{t+4}\Big)}{(t+2)^2-(t+4)} \\
&= \lim_{t\to 0^+} \frac{t\Big(5t\sin^2(\ln t^2)+4\Big)\Big(t+2+\sqrt{t+4}\Big)}{t^2+4t+4-t-4} \\
&= \lim_{t\to 0^+} \frac{t\Big(5t\sin^2(\ln t^2)+4\Big)\Big(t+2+\sqrt{t+4}\Big)}{t^2+3t} \\
&= \lim_{t\to 0^+} \frac{\Big(5t\sin^2(\ln t^2)+4\Big)\Big(t+2+\sqrt{t+4}\Big)}{t+3}
\end{aligned}$$

We can't proceed so we need some special trick: we know $0 \leq \sin^2(\ln t^2) \leq 1$

$$\frac{4\Big(t+2+\sqrt{t+4}\Big)}{t+3} \leq \frac{\Big(5t\sin^2(\ln t^2)+4\Big)\Big(t+2+\sqrt{t+4}\Big)}{t+3} \leq \frac{(5t+4)\Big(t+2+\sqrt{t+4}\Big)}{t+3}$$

Hence

$$\lim_{t\to 0^+} \frac{4\Big(t+2+\sqrt{t+4}\Big)}{t+3} \leq \lim_{t\to 0^+} \frac{\Big(5t\sin^2(\ln t^2)+4\Big)\Big(t+2+\sqrt{t+4}\Big)}{t+3} \leq \lim_{t\to 0^+} \frac{(5t+4)\Big(t+2+\sqrt{t+4}\Big)}{t+3}$$

which gives

$$\frac{4\Big(2+\sqrt{4}\Big)}{3} \leq \lim_{t\to 0^+} \frac{\Big(5t\sin^2(\ln t^2)+4\Big)\Big(t+2+\sqrt{t+4}\Big)}{t+3} \leq \frac{4\Big(2+\sqrt{4}\Big)}{3}$$

By squeeze theorem

$$\lim_{t\to 0^+} \frac{\Big(5t\sin^2(\ln t^2)+4\Big)\Big(t+2+\sqrt{t+4}\Big)}{t+3} = \frac{16}{3}$$

22. $\lim\limits_{x\to\infty} x\ln\left(\dfrac{x+1}{x-1}\right)$

$$\lim_{x\to\infty} x\ln\left(\frac{x+1}{x-1}\right) = \lim_{x\to\infty} \frac{\ln(x+1)-\ln(x-1)}{\frac{1}{x}} \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{x+1}-\frac{1}{x-1}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{2x^2}{(x+1)(x-1)} = 2$$

23. $\lim\limits_{x \to 0^+} x \ln\big(\sin x\big)$

$$\lim_{x \to 0^+} x \ln\big(\sin x\big) = \lim_{x \to 0^+} \frac{\ln\big(\sin x\big)}{\frac{1}{x}} \overset{LH}{=} \lim_{x \to 0^+} \frac{\frac{\cos x}{\sin x}}{\frac{-1}{x^2}} = \lim_{x \to 0^+} \frac{-x^2 \cos x}{\sin x} = \frac{\lim\limits_{x \to 0^+} -x \cos x}{\lim\limits_{x \to 0^+} \dfrac{\sin x}{x}} = \frac{0 \cdot 1}{1} = 0$$

24. $\lim\limits_{x \to 0^+} x^{\sin x}$

$$\ln\Big(\lim_{x \to 0^+} x^{\sin x}\Big) = \lim_{x \to 0^+} \ln x^{\sin x} = \lim_{x \to 0^+} \sin x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\csc x} \overset{LH}{=} \lim_{x \to 0^+} \frac{x^{-1}}{-\csc x \cot x} = \lim_{x \to 0^+} \frac{-\sin^2 x}{x \cos x}$$

$$= -\lim_{x \to 0^+} \frac{\sin x}{x} \tan x = -\Big(\lim_{x \to 0^+} \frac{\sin x}{x}\Big)\Big(\lim_{x \to 0^+} \tan x\Big) = -1 \cdot 1 \cdot 0 = 0 \quad \implies \quad \lim_{x \to 0^+} x^{\sin x} = e^0 = 1$$

2.7 Differentiation

1. $\frac{\mathrm{d}}{\mathrm{d}x}5x^5$
2. $\frac{\mathrm{d}}{\mathrm{d}x}2x^2$
3. $\frac{\mathrm{d}}{\mathrm{d}x}4\sqrt{x}-2x-7$
4. $\frac{\mathrm{d}}{\mathrm{d}x}x^2-4x^6$
5. $\frac{\mathrm{d}}{\mathrm{d}x}7x+\sqrt{x}$
6. $\frac{\mathrm{d}}{\mathrm{d}x}4\sqrt{x}-x^{\frac{3}{2}}$
7. $\frac{\mathrm{d}}{\mathrm{d}x}4\sqrt{x}+\frac{1}{4\sqrt{x}}$
8. $\frac{\mathrm{d}}{\mathrm{d}x}x+x^2+x^3+x^4$
9. $\frac{\mathrm{d}}{\mathrm{d}x}(3+2\sqrt{x})^2$
10. $\frac{\mathrm{d}}{\mathrm{d}x}2\sqrt{x}(7x-x^2)$
11. $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x+x^2}{\sqrt{x}}$
12. $\frac{\mathrm{d}}{\mathrm{d}x}\frac{(x+2)(2x-3)}{4x^5}$
13. $\frac{\mathrm{d}}{\mathrm{d}x}2x^3-\sqrt{x}+\frac{x^2+2x}{x^2}, x>0$
14. $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^{-\sqrt{2}}}$
15. $\frac{\mathrm{d}}{\mathrm{d}x}\sin(\sqrt{x})+x^2e^x$
16. $\frac{d^3}{\mathrm{d}x^3}\Big(e^{3x}+\sin(2x)+\frac{x^5}{20}+13x^3\Big)$
17. $\frac{\mathrm{d}}{\mathrm{d}x}x^3\ln x$
18. $\frac{\mathrm{d}}{\mathrm{d}x}e^x(x+1)$
19. $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}\ln x$
20. $\frac{\mathrm{d}}{\mathrm{d}x}e^x\cos x$
21. $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}\sin x$
22. $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3}{\ln x}$
23. $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x+1}{x^2+1}$
24. $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sqrt{x}}{\ln x}$
25. $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\cos x}{\sin x}$
26. $\frac{\mathrm{d}}{\mathrm{d}x}(2x+1)^7$
27. $\frac{\mathrm{d}}{\mathrm{d}x}(x^2+1)^2$
28. $\frac{\mathrm{d}}{\mathrm{d}x}\cos(x^3+1)$
29. $\frac{\mathrm{d}}{\mathrm{d}x}e^{x^2}$
30. $\frac{\mathrm{d}}{\mathrm{d}x}(\tan(x^2-1))^3$
31. $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}(2x+1)^4$
32. $\frac{\mathrm{d}}{\mathrm{d}x}e^{2x}(x^2+1)^2$
33. $\frac{\mathrm{d}}{\mathrm{d}x}\frac{(\tan(x^2-1))^3}{x}$
34. $\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1)\Big(\frac{x^3+1}{x}\Big)^{1/2}$
35. Given $e^y+\sin y=3x^3-2x$, find $\frac{dy}{\mathrm{d}x}$
36. $\frac{d^3}{\mathrm{d}x^3}\ln(1-\sin x)$
37. $\frac{d^3}{\mathrm{d}x^3}\Big(x^e+\sin(x^2\cos x)+\tan^{-1}\big(\sqrt{x^2+1}\big)+\frac{\ln(7e^{3/2})}{42}\Big)$
38. $\frac{\mathrm{d}}{\mathrm{d}x}x^{e^x}$

2.8 Differentiation solution

1. $\frac{\mathrm{d}}{\mathrm{d}x}5x^5$ $25x^4$
2. $\frac{\mathrm{d}}{\mathrm{d}x}2x^2$ $4x$
3. $\frac{\mathrm{d}}{\mathrm{d}x}4\sqrt{x}-2x-7$ $4x^{-\frac{1}{2}}-2$
4. $\frac{\mathrm{d}}{\mathrm{d}x}x^2-4x^6$ $2x-24x^5$
5. $\frac{\mathrm{d}}{\mathrm{d}x}7x+\sqrt{x}$ $7+\frac{1}{2\sqrt{x}}$
6. $\frac{\mathrm{d}}{\mathrm{d}x}4\sqrt{x}-x^{\frac{3}{2}}$ $2x^{-\frac{1}{2}}-\frac{1}{3}x^{-\frac{2}{3}}$
7. $\frac{\mathrm{d}}{\mathrm{d}x}4\sqrt{x}+\frac{1}{4\sqrt{x}}$ $\frac{2}{\sqrt{x}}-\frac{1}{8}x^{-\frac{3}{2}}$
8. $\frac{\mathrm{d}}{\mathrm{d}x}x+x^2+x^3+x^4$ $1+2x+3x^2+4x^3$
9. $\frac{\mathrm{d}}{\mathrm{d}x}(3+2\sqrt{x})^2$ $\frac{\mathrm{d}}{\mathrm{d}x}9+6\sqrt{x}+4x=6x^{-\frac{1}{2}}+4$
10. $\frac{\mathrm{d}}{\mathrm{d}x}2\sqrt{x}(7x-x^2)$ $\frac{\mathrm{d}}{\mathrm{d}x}14x^{\frac{3}{2}}-2x^{\frac{5}{2}}=21x^{\frac{1}{2}}-5x^{\frac{3}{2}}$
11. $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x+x^2}{\sqrt{x}}$ $\frac{1}{2}x^{-\frac{1}{2}}+\frac{3}{2}x^{\frac{1}{2}}$
12. $\frac{\mathrm{d}}{\mathrm{d}x}\frac{(x+2)(2x-3)}{4x^5}$ $-\frac{3}{2}x^{-4}-x^{-5}+\frac{15}{2}x^{-6}$
13. $\frac{\mathrm{d}}{\mathrm{d}x}2x^3-\sqrt{x}+\frac{x^2+2x}{x^2}, x>0$ $6x^2+\frac{1}{2\sqrt{x}}-\frac{2}{x^2}$
14. $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^{-\sqrt{2}}}$ $-\frac{\sqrt{2}}{2}x^{\frac{\sqrt{2}}{2}-1}$
15. $\frac{\mathrm{d}}{\mathrm{d}x}\sin(\sqrt{x})+x^2e^x$ $\cos(x^{1/2})\frac{1}{2}x^{-1/2}+2xe^x+x^2e^x$
16. $\frac{d^3}{\mathrm{d}x^3}\left(e^{3x}+\sin(2x)+\frac{x^5}{20}+13x^3\right)$ $27e^{3x}-8\cos 2x+3x^2+78$
17. $\frac{\mathrm{d}}{\mathrm{d}x}x^3\ln x$ $3x^2\ln x+x^2$
18. $\frac{\mathrm{d}}{\mathrm{d}x}e^x(x+1)$ $e^x(x+1)+e^x=e^x(x+2)$
19. $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}\ln x$ $\frac{\ln x}{2\sqrt{x}}+\frac{1}{\sqrt{x}}$
20. $\frac{\mathrm{d}}{\mathrm{d}x}e^x\cos x$ $e^x\cos x-e^x\sin x=e^x(\cos x-\sin x)$
21. $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}\sin x$ $\frac{\sin x}{2\sqrt{x}}+\sqrt{x}\cos x$
22. $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3}{\ln x}$ $\frac{3x^2\ln x-x^3(1/x)}{(\ln x)^2}=\frac{x^2(3\ln x-1)}{(\ln x)^2}$
23. $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x+1}{x^2+1}$ $\frac{1\cdot(x^2+1)-(x+1)2x}{(x^2+1)^2}=\frac{-x^2-2x+1}{(x^2+1)^2}$
24. $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sqrt{x}}{\ln x}$ $\frac{\frac{1}{2\sqrt{x}}\ln x-(x+1)\sqrt{x}}{(\ln x)^2}$
25. $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\cos x}{\sin x}$ $\frac{-\sin x\sin x-\cos x\cos x}{\sin^2 x}=\frac{-1}{\sin^2 x}=-\csc^2 x$

26. $\dfrac{\mathrm{d}}{\mathrm{d}x}(2x+1)^7$ $\quad = 7(2x+1)^6 2 = 14(2x+1)^6$

27. $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^2+1)^2$ $\quad 2(x^2+1)2x = 4x(x^2+1)$

28. $\dfrac{\mathrm{d}}{\mathrm{d}x}\cos(x^3+1)$ $\quad -\sin(x^3+1)3x^2$

29. $\dfrac{\mathrm{d}}{\mathrm{d}x}e^{x^2}$ $\quad e^{x^2}2x$

30. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan(x^2-1))^3$ $\quad 3(\tan(x^2-1))^2\sec^2(x^2-1)2x$

31. $\dfrac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}(2x+1)^4$ $\quad \dfrac{(2x+1)^3}{2\sqrt{x}}(18x+1)$

32. $\dfrac{\mathrm{d}}{\mathrm{d}x}e^{2x}(x^2+1)^2$ $\quad 2e^{2x}(x^2+1)(x+1)^2$

33. $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{(\tan(x^2-1))^3}{x}$ $\quad \dfrac{6x(\tan(x^2-1))^2\sec^2(x^2-1)+(\tan(x^2-1))^3}{x^2}$

34. $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^2-1)\Big(\dfrac{x^3+1}{x}\Big)^{1/2}$ $\quad \Big(\dfrac{x}{x^3+1}\Big)^{1/2}\dfrac{6x^5-2x^3+3x^2+1}{2x^2}$

35. Given $e^y+\sin y = 3x^3-2x$, find $\dfrac{dy}{\mathrm{d}x}$

$$e^y\frac{dy}{\mathrm{d}x}+\cos y\frac{dy}{\mathrm{d}x} = 9x^2-2 \quad\Longleftrightarrow\quad \frac{dy}{\mathrm{d}x}(e^y+\cos y) = 9x^2-2 \quad\Longleftrightarrow\quad \frac{dy}{\mathrm{d}x} = \frac{9x^2-2}{e^y+\cos y}$$

36. $\dfrac{d^3}{\mathrm{d}x^3}\ln(1-\sin x)$ $\quad \dfrac{-\cos x}{(1-\sin x)^2}$

$$\begin{aligned}
f' &= -\frac{\cos x}{1-\sin x} && \text{chain rule}\\
f'' &= \frac{(1-\sin x)\sin x-(-\cos x)(-\cos x)}{(1-\sin x)^2} && \text{quotient rule}\\
&= \frac{\sin x-\sin^2 x-\cos^2 x}{(1-\sin x)^2}\\
&= \frac{\sin x-1}{(1-\sin x)^2}\\
&= \frac{-1}{1-\sin x}\\
f'' &= \frac{(-1)(-1)(-\cos x)}{(1-\sin x)^2} && \text{quotient rule}\\
&= \frac{-\cos x}{(1-\sin x)^2}
\end{aligned}$$

37. $\dfrac{d^3}{\mathrm{d}x^3}\Big(x^e+\sin(x^2\cos x)+\tan^{-1}\big(\sqrt{x^2+1}\big)+\dfrac{\ln(7e^{3/2})}{42}\Big)$

$$\begin{aligned}
f' &= ex^{e-1}+\cos(x^2\cos x)\Big(2x\cos x-x^2\sin x\Big)+\frac{1}{1+(\sqrt{x^2+1})^2}\frac{1}{2}(1+x^2)^{-1/2}2x\\
&= ex^{e-1}+\cos(x^2\cos x)\Big(2x\cos x-x^2\sin x\Big)+\frac{x}{2+x^2}\frac{1}{\sqrt{1+x^2}}
\end{aligned}$$

38. $\dfrac{\mathrm{d}}{\mathrm{d}x}x^{e^x}$

$$y = x^{e^x} \quad\Longleftrightarrow\quad \ln y = e^x\ln x \quad \frac{y'}{y} = \frac{e^x}{x}+e^x\ln x \quad\Longrightarrow\quad y' = y\Big(\frac{e^x}{x}+e^x\ln x\Big) = e^x\ln x\Big(\frac{e^x}{x}+e^x\ln x\Big)$$

2.9 Feynman's differentiation trick

$$f = k \cdot u^a \cdot v^b \cdot w^c \cdots \qquad \Longrightarrow \qquad f' = f \cdot \left[a\frac{u'}{u} + b\frac{v'}{v} + c\frac{w'}{w} + \dots \right].$$

1. $\dfrac{\mathrm{d}}{dx}\dfrac{e^x(x^4+1)^x}{x}$ $\qquad f' = \left(1 + \ln(x^4+1) + \dfrac{4x^4}{x^4+1} - \dfrac{1}{x}\right)\dfrac{e^x(x^4+1)^x}{x}$

2. $\dfrac{\mathrm{d}}{dx}\dfrac{x^{\sin x}\cos x}{e^{3x}+5}$ $\qquad f' = \left(\cos x \ln x + \dfrac{\sin x}{x} - \tan x - \dfrac{e^{3x}}{e^{3x}+5}\right)\dfrac{x^{\sin x}\cos x}{e^{3x}+5}$

3. $\dfrac{\mathrm{d}}{dx} x^\pi e^{\cos 2x}(\sec x)^x$ $\qquad f' = \left(\dfrac{\pi}{x} - 2\sin 2x + \ln \sec x + x\tan x\right) x^\pi e^{\cos 2x}(\sec x)^x$

4. $\dfrac{\mathrm{d}}{dx}\dfrac{(1+x^2)^3(4x-5)^6}{(7x+8)^9(10x-11)^{12}}$ $\qquad f' = \dfrac{(1+x^2)^3(4x-5)^6}{(7x+8)^9(10x-11)^{12}}\left[3\dfrac{2x}{1+x^2} + 6\dfrac{4}{4x-5} - 9\dfrac{7}{7x+8} - 12\dfrac{10}{10x-11}\right]$

5. $\dfrac{\mathrm{d}}{dx}\dfrac{7(2+3x)^2(x^2-x)^3}{(x+4x^3)^{\frac{1}{2}}(6x)^{\frac{3}{2}}}$ $\qquad f' = \dfrac{7(2+3x)^2(x^2-x)^3}{(x+4x^3)^{\frac{1}{2}}(6x)^{\frac{3}{2}}}\left[2\dfrac{3}{2+3x} + 3\dfrac{2x-1}{x^2-x} - \dfrac{1}{2}\dfrac{1+12x^2}{x+4x^3} + \dfrac{3}{2}\dfrac{6}{x}\right]$

6. $\dfrac{\mathrm{d}}{dx}\big(\sec x + \tan x\big)^{\cos x}$ $\qquad \big(\sec x + \tan x\big)^{\cos x}\Big(1 - \sin x \ln\big(\sec x + \tan x\big)\Big)$

2.10 Integration

2.10.1 Polynomial

1. $\int 1 \mathrm{d}x$ $\qquad x + C$

2. $\int x + x^2 + x^3 \mathrm{d}x$ $\qquad \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + C$

3. $\int x^3 - 6x^2 + 12 \,\mathrm{d}x$ $\qquad \frac{1}{4}x^4 - 2x^3 + 12x + C$

4. $\int \sqrt{x} - \frac{1}{x^2} + 4\sqrt{x^3} \,\mathrm{d}x$ $\qquad \frac{3}{2}x^{\frac{2}{3}} + x^{-1} + \frac{8}{5}x^{\frac{5}{2}} + C$

5. $\int 2\sqrt{x} + \frac{1}{2\sqrt{x}} \mathrm{d}x$ $\qquad \frac{4}{3}x^{\frac{3}{2}} + \sqrt{x} + C$

6. $\int 14x^{\frac{3}{4}} - \frac{3}{2x^4} \mathrm{d}x$ $\qquad 8x^{\frac{7}{4}} + \frac{1}{2}x^{-3} + C$

7. $\int 6\sqrt{x^3} - \frac{1}{2x^5} \mathrm{d}x$ $\qquad \frac{12}{5}x^{\frac{5}{2}} - \frac{1}{8}x^{-4} + C$

8. $\int \frac{2 + 5x}{3x^3} \mathrm{d}x$ $\qquad -\frac{1}{3x^2} - \frac{5}{3x} + C$

9. $\int \frac{\sqrt{x}(4 - x)}{2x^2} \mathrm{d}x$ $\qquad -\sqrt{x} - \frac{4}{\sqrt{x}} + C$

10. $\int \frac{(x+1)(2x-1)}{2x^5} \mathrm{d}x$ $\qquad \frac{-1}{2x^2} - \frac{1}{6x^3} + \frac{1}{8x^4} + C$

11. $\int 2\sqrt{x}\Big(\frac{5}{x} + x^2\Big) \mathrm{d}x$ $\qquad 20\sqrt{x} + \frac{4}{7}x^{\frac{7}{2}} + C$

12. $\int \frac{(1 + 2\sqrt{x})^2}{2x^3} \mathrm{d}x$ $\qquad -\frac{1}{4}x^{-2} - \frac{4}{3}x^{-\frac{3}{2}} - 2x^{-1} + C$

13. $\int (3x+1)^2 \mathrm{d}x$ $\qquad \frac{1}{9}(3x+1)^3 + C$

14. $\int 4(2x+1)^5 \mathrm{d}x$ $\qquad \frac{1}{3}(2x+1)^6 + C$

15. $\int 6(4x-3)^{\frac{1}{2}} \mathrm{d}x$ $\qquad (4x-3)^{\frac{3}{2}} + C$

16. $\int \frac{6}{\sqrt{3x+1}} \mathrm{d}x$ $\qquad 4\sqrt{3x+1} + C$

17. $\int \frac{4}{(3x-1)^2} \mathrm{d}x$ $\qquad -\frac{4}{3}\frac{1}{(3x-1)} + C$

18. $\int 15(1-2x)^{\frac{3}{2}} \mathrm{d}x$ $\qquad -3(1-2x)^{\frac{5}{2}} + C$

19. $\int_1^4 \frac{2}{\sqrt{x}} \mathrm{d}x$ $\qquad 4$

20. $\int_0^2 (2x-1)(3x-4) \mathrm{d}x$ $\qquad 2$

21. $\int_1^3 x^2 + \frac{14}{x^2} \mathrm{d}x$ $\qquad 18$

22. $\int_1^2 \frac{(3x^2-2)^2}{x^2} \mathrm{d}x$ $\qquad 11$

23. $\int_1^5 2x - \frac{15}{x^2} \mathrm{d}x$ $\qquad 12$

24. $\int_1^4 \sqrt{x}(5x-3) \mathrm{d}x$ $\qquad 48$

25. $\displaystyle\int_1^2 \frac{2x^5+3}{x^2}\mathrm{d}x$ $\quad 9$

26. $\displaystyle\int_1^5 3\sqrt{x}-\frac{1}{\sqrt{x}}\mathrm{d}x\mathrm{d}x$ $\quad 8\sqrt{5}$

27. $\displaystyle\int_0^1 (2+\sqrt{x})^2\mathrm{d}x$ $\quad \frac{43}{6}$

28. $\displaystyle\int_3^4 3\sqrt{x}-\frac{4}{\sqrt{x}}\mathrm{d}x = a\sqrt{3}$, find a $\quad a=2$

29. $\displaystyle\int_1^3 2x^2+3x+a\mathrm{d}x = \frac{4}{3}$, find a $\quad k=-14$

30. $\displaystyle\int_1^2 (3+2\sqrt{x})^2\mathrm{d}x = a+b\sqrt{2}$, find a,b $\quad a=7, b=16$

2.10.2 Fraction

1. $\displaystyle\int \frac{x-3}{x-8}\,\mathrm{d}x$ $\quad x+5\ln|x-8|+C$

2. $\displaystyle\int \frac{2x-3}{x+2}\,\mathrm{d}x$ $\quad 2x-7\ln|x+2|+C$

3. $\displaystyle\int \frac{3x-1}{x-1}\,\mathrm{d}x$ $\quad 3x+2x\ln|x-1|+C$

4. $\displaystyle\int \frac{3-x}{x-1}\,\mathrm{d}x$ $\quad -x+2\ln|x-1|+C$

5. $\displaystyle\int \frac{x+3}{2x-1}\,\mathrm{d}x$ $\quad \frac{1}{2}x+\frac{7}{4}\ln|2x-1|+C$

6. $\displaystyle\int \frac{3x+1}{2x-1}\,\mathrm{d}x$ $\quad \frac{3}{2}x+\frac{5}{4}\ln|2x-1|+C$

7. $\displaystyle\int \frac{7x+3}{7x-4}\,\mathrm{d}x$ $\quad x+\ln|7x-4|+C$

8. $\displaystyle\int \frac{x+a}{x-a}\,\mathrm{d}x$ $\quad x+2\ln|x-a|+C$

9. $\displaystyle\int \frac{ax+b}{ax-b}\,\mathrm{d}x$ $\quad x+2\frac{b}{a}\ln|ax-b|+C$

10. $\displaystyle\int \frac{ax+b}{cx+d}\,\mathrm{d}x$ $\quad \frac{acx+(bc-ad)\ln|cx+d|}{c^2}+C$

2.10.3 Trigonometric

1. Show $\displaystyle\int \sec\theta\,\mathrm{d}\theta = \ln|\sec\theta+\tan\theta|+C$ by considering $\displaystyle\int \sec\theta\,\mathrm{d}\theta = \int \sec\theta\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\,\mathrm{d}\theta$.

2. Show $\displaystyle\int \sec\theta\,\mathrm{d}\theta = \frac{1}{2}\ln\frac{1+\sin\theta}{1-\sin\theta}+C$ by considering $\displaystyle\int \sec\theta\,\mathrm{d}\theta = \int \frac{\cos\theta}{1-\sin^2\theta}\,\mathrm{d}\theta$ with substitution $u=\sin\theta$ and partial fraction.

3. $\displaystyle\int 2\tan^2 x\,\mathrm{d}x$ $\quad 2\tan x-2x+C$

4. $\displaystyle\int \frac{2\sin x}{\cos^2 x}\,\mathrm{d}x$ $\quad 2\sec x+C$

5. $\displaystyle\int 5\cot^2 x\,\mathrm{d}x$ $\quad -5\cot x-5x+C$

6. $\displaystyle\int \sin x(1+\sec^2 x)\,\mathrm{d}x$ $\quad \sec x-\cos x+C$

7. $\displaystyle\int 2+2\tan^2 x\,\mathrm{d}x$ $\quad 2\tan x+C$

8. $\int 10\sin 2x \,\mathrm{d}x$ $\quad -5\cos 2x + C$

9. $\int 4\cos 3x \,\mathrm{d}x$ $\quad \frac{4}{3}\sin 3x + C$

10. $\int 6\cos 2x \,\mathrm{d}x$ $\quad 3\sin 2x + C$

11. $\int 8\cos 2x - 12\sin 3x \,\mathrm{d}x$ $\quad 4\sin 2x + 4\cos 3x + C$

12. $\int \cos\frac{x}{4} + \sin\frac{x}{2} \,\mathrm{d}x$ $\quad 4\sin\frac{x}{4} - 2\cos\frac{x}{2} + C$

13. $\int \cos(5-2x) \,\mathrm{d}x$ $\quad -\frac{1}{2}\sin(5-2x) + C$

14. $\int \frac{2}{\cos^2 x} \,\mathrm{d}x$ $\quad 2\tan x + C$

15. $\int \tan 2x \sec 2x \,\mathrm{d}x$ $\quad \frac{1}{2}\sec 2x + C$

16. $\int \csc^2(3x+1) \,\mathrm{d}x$ $\quad -\frac{1}{3}\cot(3x+1) + C$

17. $\int 12\sec^2(2x+3) \,\mathrm{d}x$ $\quad 6\tan(2x+3) + C$

18. $\int \tan 3x \,\mathrm{d}x$ $\quad \frac{1}{3}\ln|\sec 3x| + C$

19. $\int \cot kx \,\mathrm{d}x$ $\quad \frac{1}{k}\ln|\sin kx| + C$

20. $\int \frac{1+\sin x}{\cos^2 x} \,\mathrm{d}x$ $\quad \sec x + \tan x + C$

21. $\int \tan^2 x \,\mathrm{d}x$ $\quad \tan x - x + C$

22. $\int \sec 2x \sec x \,\mathrm{d}x$ $\quad -2\cos x + C$

2.10.4 Substitution

1. $\int e^x + e^{2x} + e^{3x} \,\mathrm{d}x$ $\quad e^x + \frac{1}{2}e^{2x} + \frac{1}{3}e^{3x} + C$

2. $\int 2^x \,\mathrm{d}x$ $\quad \frac{1}{\ln 2}2^x + C$

3. $\int \frac{1}{x+1} + \frac{1}{2x-1} + \frac{1}{2-x} \,\mathrm{d}\theta$ $\quad \ln|x+1| + \frac{1}{2}\ln|2x-1| - \ln|2-x| + C$

4. $\int 5x(x^2-1)^4 \,\mathrm{d}x$ $\quad \frac{1}{2}(x^2-1)^5 + C$

5. $\int \frac{4x}{\sqrt{x^2-7}} \,\mathrm{d}x$ $\quad 4\sqrt{x^2-7} + C$

6. $\int x^3(1-4x^4)^{-\frac{1}{2}} \,\mathrm{d}x$ $\quad -\frac{1}{6}(1-4x^4)^{\frac{1}{2}} + C$

7. $\int \frac{\ln x}{x} \,\mathrm{d}x$ $\quad \frac{1}{2}(\ln x)^2 + C$

8. $\int \frac{x}{9x^2+1} \,\mathrm{d}x$ $\quad \frac{1}{18}\ln(9x^2+1) + C$

9. $\int \frac{3^x}{3^x+1} \,\mathrm{d}x$ $\quad \frac{\ln(3^x+1)}{\ln 3} + C$

10. $\int \frac{x^3}{x^4+2} \,\mathrm{d}x$ $\quad \frac{1}{4}\ln(x^4+2) + C$

11. $\displaystyle\int \frac{x^2}{4-x^3}\,\mathrm{d}x$ $\quad -\frac{1}{3}\ln|4-x^3|+C$

12. $\displaystyle\int \frac{4x}{x^2-1}\,\mathrm{d}x$ $\quad 2\ln|x^2-1|+C$

13. $\displaystyle\int \frac{3x^2}{1+x^3}\,\mathrm{d}x$ $\quad \ln|1+x^3|+C$

14. $\displaystyle\int \frac{4x}{x^2-10}\,\mathrm{d}x$ $\quad 2\ln|x^2-10|+C$

15. $\displaystyle\int \frac{2x+6}{x^2+6x+1}\,\mathrm{d}x$ $\quad \ln|x^2+6x+1|+C$

16. $\displaystyle\int \frac{2ax+b}{ax^2+bx+c}\,\mathrm{d}x$ $\quad \ln|ax^2+bx+c|+C$

17. $\displaystyle\int \frac{4^x}{4^x+4}\,\mathrm{d}x$ $\quad \frac{\ln(4^x+4)}{\ln 4}+C$

18. $\displaystyle\int \frac{1}{x\ln x}\,\mathrm{d}x$ $\quad \ln\big|\ln|x|\big|+C$

19. $\displaystyle\int 3xe^{x^2}\,\mathrm{d}x$ $\quad \frac{3}{2}e^{x^2}+C$

20. $\displaystyle\int \frac{3e^{2x}}{e^{2x}-1}\,\mathrm{d}x$ $\quad \frac{3}{2}\ln|e^{2x}-1|+C$

21. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$ $\quad 2e^{\sqrt{x}}+C$

22. $\displaystyle\int (2x+1)(x^2+x+1)\,\mathrm{d}x$ $\quad \frac{1}{2}(x^2+x+1)^2+C$

23. $\displaystyle\int \frac{4x}{3}(2x^2-1)\,\mathrm{d}x$ $\quad \frac{1}{3}(2x^2-1)^2+C$

24. $\displaystyle\int 3x^2(4-2x^3)^{3/2}\,\mathrm{d}x$ $\quad -\frac{1}{5}(4-x2^3)^{5/2}+C$

25. $\displaystyle\int \frac{\sqrt{\sqrt{x}+1}}{\sqrt{x}}\,\mathrm{d}x$ $\quad \frac{4}{3}\left(\sqrt{x}+1\right)^{\frac{3}{2}}+C$

26. $\displaystyle\int \frac{1}{\sqrt{x}\sqrt{\sqrt{x}+1}}\,\mathrm{d}x$ $\quad 4\sqrt{\sqrt{x}+1}+C$

27. $\displaystyle\int \sin 2x\,\mathrm{d}x$ $\quad -\frac{1}{2}\cos 2x+C$

28. $\displaystyle\int 4\sin^3 x\cos x\,\mathrm{d}x$ $\quad \sin^4 x+C$

29. $\displaystyle\int \sin x\cos^2 x\,\mathrm{d}x$ $\quad -\frac{1}{3}\cos^3 x+C$

30. $\displaystyle\int \tan^4 x\sec^2 x\,\mathrm{d}x$ $\quad \frac{1}{5}\tan^5 x+C$

31. $\displaystyle\int \frac{3\sec^2 x}{\tan x}\,\mathrm{d}x$ $\quad 3\ln|\tan x|+C$

32. $\displaystyle\int \cos x\sin x\,\mathrm{d}x$ $\quad \frac{\sin^2 x}{2}+C,\ \frac{-\cos^2 x}{2}+C,$

33. $\displaystyle\int \sec^2(1+\tan x)\,\mathrm{d}x$ $\quad \frac{(1+\tan x)^2}{2}+C$

34. $\displaystyle\int \frac{\cos x}{\sqrt{\sin x}}\,\mathrm{d}x$ $\quad 2\sqrt{\sin x}+C$

35. $\displaystyle\int \frac{1}{\cos^2 x\tan^4 x}\,\mathrm{d}x$ $\quad -\frac{1}{3\tan^3 x}+C$

36. $\displaystyle\int (3x^2+1)\sin(x^3+x)\,\mathrm{d}x$ $\qquad -\cos(x^3+x)+C$

37. $\displaystyle\int \frac{\cos x}{\sqrt{\sin^3 x}}\,\mathrm{d}x$ $\qquad \displaystyle -\frac{2}{\sqrt{\sin x}}+C$

38. $\displaystyle\int \cos x\sqrt{\sin x}\,\mathrm{d}x$ $\qquad \displaystyle \frac{2}{3}\sqrt{\sin^3 x}+C$

39. $\displaystyle\int \frac{\sin x-\cos x}{\sin x+\cos x}\,\mathrm{d}x$ $\qquad -\ln|\sin x+\cos x|+C$

40. $\displaystyle\int \frac{\cos(\ln x)}{x}\,\mathrm{d}x$ $\qquad \sin(\ln x)+C$

41. $\displaystyle\int e^{\sin x}\cos x\,\mathrm{d}x$ $\qquad e^{\sin x}+C$

42. $\displaystyle\int e^{x}\sin(e^x)\,\mathrm{d}x$ $\qquad -\cos(e^x)+C$

43. $\displaystyle\int e^{x+e^x}\,\mathrm{d}x$ $\qquad \displaystyle u=e^x,\int e^{x+e^x}\,\mathrm{d}x=\int e^x e^{e^x}\,\mathrm{d}x=e^{e^x}+C$

44. $\displaystyle\int \frac{1}{1+x^2}\,\mathrm{d}x$ using $x=\tan\theta$
Let $x=\tan\theta \implies \mathrm{d}x=\sec^2\theta\mathrm{d}\theta$

$$\int\frac{1}{1+x^2}\,\mathrm{d}x \;=\; \int\frac{1}{1+\tan^2\theta}\sec^2\theta\mathrm{d}\theta \;=\; \int\frac{\sec^2\theta}{\sec^2\theta}\,\mathrm{d}\theta \;=\; \int \mathrm{d}\theta \;=\; \theta \;=\; \tan^{-1}x$$

45. $\displaystyle\int \frac{\sin^2 x}{\cos^2 x}\,\mathrm{d}x$
Let $t=\tan x$ thus $\mathrm{d}t=\sec^2 x\mathrm{d}x$

$$\begin{aligned}\int\frac{\sin^2 x}{\cos^2 x}\,\mathrm{d}x &= \int \sin^2 x\sec^2 x\,\mathrm{d}x \;=\; \int\sin^2 x\,\mathrm{d}t \;=\; \int(1-\cos^2 x)\,\mathrm{d}t \;=\; \int\Big(1-\frac{1}{\sec^2 x}\Big)\,\mathrm{d}t\\ &= \int\Big(1-\frac{1}{1+\tan^2 x}\Big)\,\mathrm{d}t \;=\; \int\Big(1-\frac{1}{1+t^2}\Big)\,\mathrm{d}t \;=\; \int\mathrm{d}t-\int\frac{\mathrm{d}t}{1+t^2}\\ &= t-\tan^{-1}t+C \;=\; \tan x - x + C\end{aligned}$$

46. $\displaystyle\int \frac{\sin 2x}{\cos^2 x}\,\mathrm{d}x$

$$\int\frac{\sin 2x}{\cos^2 x}\,\mathrm{d}x = 2\int\frac{\sin x\cos x}{\cos^2 x}\,\mathrm{d}x = 2\int\frac{\sin x}{\cos x}\,\mathrm{d}x$$

Let $\cos x = t \implies \sin x\mathrm{d}x=\mathrm{d}t$

$$\int\frac{\sin 2x}{\cos^2 x}\,\mathrm{d}x = 2\int\frac{1}{t}\,\mathrm{d}t = 2\ln|t|+C = 2\ln|\cos x|+C$$

47. $\displaystyle\int \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}\,\mathrm{d}x$
Let $x=t^2$, then $\mathrm{d}x=2t\mathrm{d}t$

$$\int\frac{1}{\sqrt{x}(\sqrt{x}+1)^2}\,\mathrm{d}x \;=\; \int\frac{2t\,\mathrm{d}t}{t(t+1)^2} \;=\; \int\frac{2\mathrm{d}t}{(t+1)^2} \;=\; \frac{-2}{t+1}+C \;=\; \frac{-2}{\sqrt{x}+1}+C$$

48. $\displaystyle\int \frac{1}{\sqrt[3]{x}(\sqrt[6]{x}+1)^5}\,\mathrm{d}x$

$$\begin{aligned}u=\sqrt[6]{x}+1 \implies \int\frac{6(u-1)^5}{(u-1)^2u^5}\,\mathrm{d}u &= 6\int\frac{(u-1)^3}{u^5}\,\mathrm{d}u\\ &= 6\int\frac{u^3-3u^2+3u-1}{u^5}\,\mathrm{d}u\\ &= 6\Big(-\frac{1}{u}+\frac{3}{2u^2}-\frac{1}{u^3}+\frac{1}{4u^4}\Big)+C\\ &= -\frac{6}{\sqrt[6]{x}+1}+\frac{9}{(\sqrt[6]{x}+1)^2}-\frac{6}{(\sqrt[6]{x}+1)^3}+\frac{3}{2(\sqrt[6]{x}+1)^4}+C\end{aligned}$$

49. Show $\displaystyle\int_0^\infty \frac{1}{(x+\sqrt{1+x^2})^2}\,\mathrm{d}x = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{(\sin\theta+1)^2}$ using $x=\tan\theta$
Let $x=\tan\theta$, $x=0 \implies \theta=0$ and $x=\infty \implies \theta=\pi/2$

$$\begin{aligned}\int_0^\infty \frac{1}{(x+\sqrt{1+x^2})^2}\,\mathrm{d}x &= \int_0^{\pi/2} \frac{\sec^2\theta\,\mathrm{d}\theta}{(\tan\theta+\sqrt{1+\tan^2\theta})^2} = \int_0^{\pi/2} \frac{\sec^2\theta\,\mathrm{d}\theta}{(\tan\theta+\sqrt{\sec^2\theta})^2} = \int_0^{\pi/2} \frac{\sec^2\theta\,\mathrm{d}\theta}{(\tan\theta+\sec\theta)^2}\\ &= \int_0^{\pi/2} \frac{\frac{1}{\cos^2\theta}\,\mathrm{d}\theta}{\left(\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}\right)^2} = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{(\sin\theta+1)^2}\end{aligned}$$

2.10.5 Trigonometric substitution

1. $\displaystyle\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,\mathrm{d}x$, use $x=2\sin\theta$

$x=2\sin\theta \implies \mathrm{d}x = 2\cos\theta d\theta, \quad x\in[0,\sqrt{2}] \implies \theta\in\big[0,\frac{\pi}{4}\big]$

$$\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,\mathrm{d}x = \int_0^{\pi/4} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}\,\mathrm{d}\theta = 2\int_0^{\pi/4} \frac{\sin^2\theta}{\cos\theta}\,\mathrm{d}\theta = 2\int_0^{\pi/4} \frac{1-\cos^2\theta}{\cos\theta}\,\mathrm{d}\theta = 2\int_0^{\pi/4} \sec\theta-\cos\theta\,\mathrm{d}\theta.$$

$$= 2\Big[\ln|\sec\theta+\tan\theta| - \sin\theta\Big]_0^{\pi/4} = 2\Big[\ln|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}| - \sin\frac{\pi}{4}\Big] - 2\Big[\ln|\sec 0+\tan 0| - \sin 0\Big] = 2\ln(\sqrt{2}+1)-\sqrt{2}$$

2. $\displaystyle\int_0^1 \frac{1}{(1+x^2)^2}\,\mathrm{d}x$, use $x=\tan\theta$ and $\cos 2\theta = 2\cos^2\theta - 1$.

$x=\tan\theta \implies \mathrm{d}x = \sec^2\theta\mathrm{d}\theta$, $x\in[0,1] \implies \theta\in[0,\pi/4]$

$$\int_0^1 \frac{1}{(1+x^2)^2}\,\mathrm{d}x = \int_0^{\pi/4} \frac{\sec^2\theta}{(1+\tan^2\theta)^2}\mathrm{d}\theta = \int_0^{\pi/4} \frac{\sec^2\theta}{\sec^4\theta}\mathrm{d}\theta = \int_0^{\pi/4} \frac{1}{\sec^2\theta}\mathrm{d}\theta = \int_0^{\pi/4} \cos^2\theta\mathrm{d}\theta = \int_0^{\pi/4} \frac{1+\cos 2\theta}{2}\mathrm{d}\theta$$

$$= \Big[\frac{\theta+\frac{1}{2}\sin 2\theta}{2}\Big]_0^{\pi/4} = \frac{\pi}{8}+\frac{1}{4}.$$

3. Show $\displaystyle\int_0^3 \frac{27}{(9+x^2)^2}\,\mathrm{d}x = \frac{\pi}{8}+\frac{1}{4}$, use $x=3\tan\theta$

4. Show $\displaystyle\int_{\sqrt{2}}^2 \frac{1}{x^2\sqrt{x^2-1}}\,\mathrm{d}x = \frac{\sqrt{3}-\sqrt{2}}{2}$, use $x=\sec\theta$.

5. Show $\displaystyle\int_0^1 \frac{1}{\left(1+3x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x = \frac{1}{2}$, use $x=\dfrac{1}{\sqrt{3}}\tan\theta$

6. $\displaystyle\int \frac{1}{\sqrt{2-x^2}}\,\mathrm{d}x = \frac{\pi}{4}$, use $x=\sqrt{2}\sin\theta$

7. Show $\displaystyle\int_0^1 \frac{1}{\left(4-x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x = \frac{\sqrt{3}}{12}$, use $x=2\sin\theta$,

8. Show $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{4x^2+3}\,\mathrm{d}x = \frac{\pi\sqrt{3}}{36}$, use $x=\dfrac{\sqrt{3}}{2}\tan\theta$

9. Show $\displaystyle\int_{\sqrt{2}}^2 \frac{\sqrt{x^2-1}}{x}\,\mathrm{d}x = \sqrt{3}-1-\frac{\pi}{12}$, use $x=\csc\theta$

10. Show $\displaystyle\int_0^2 \sqrt{16-x^2}\,\mathrm{d}x = \frac{4\pi+6\sqrt{3}}{3}$

11. $\displaystyle\int_0^{\pi/4} \sqrt{2-\sec^2 x}\sec^2 x\mathrm{d}x$, use $u = \tan x$.

$$\begin{aligned}
\int_0^{\pi/4} \sqrt{2-\sec^2 x}\sec^2 x\mathrm{d}x &= \int_0^{\pi/4} \sqrt{2-(1+\tan^2 x)}\sec^2 x\mathrm{d}x \\
&= \int_0^{\pi/4} \sqrt{1-\tan^2 x}\sec^2 x\mathrm{d}x \\
&= \int_0^{1} \sqrt{1-u^2}du \qquad u=\tan x, du = \sec^2 x\mathrm{d}x, x\in[0,\frac{\pi}{4}] \implies u\in[0,1] \\
&= \frac{1}{4}\text{ area of unity circle } = \frac{\pi}{4}
\end{aligned}$$

12. $\displaystyle\int \sin x\cos x\sqrt{3-\cos x}\mathrm{d}x$, use $u = 3-\cos x$

$$\int \sin x\cos x\sqrt{3-\cos x}\mathrm{d}x = \int (3-u)u^{1/2}du = 2u^{3/2} - \frac{2}{5}u^{5/2} + C = 2(3-\cos x)^{3/2} - \frac{2}{5}(3-\cos x)^{5/2} + C$$

2.10.6 Partial Fraction

1. $\displaystyle\int \frac{1}{(x+1)(x+2)}\,\mathrm{d}x$ $\qquad \ln\left|\frac{x+1}{x+2}\right| + C$
2. $\displaystyle\int \frac{4}{(x-2)(2-3x)}\,\mathrm{d}x$ $\qquad \ln\left|\frac{3x-2}{x-2}\right| + C$
3. $\displaystyle\int \frac{2}{(x-2)(x-4)}\,\mathrm{d}x$ $\qquad \ln\left|\frac{x-4}{x-2}\right| + C$
4. $\displaystyle\int \frac{3}{(2+x)(1-x)}\,\mathrm{d}x$ $\qquad \ln\left|\frac{2+x}{1-x}\right| + C$
5. $\displaystyle\int \frac{2-x}{(x+1)(2x-1)}\,\mathrm{d}x$ $\qquad \frac{1}{2}\ln|2x-1| - \ln|x+1|C$
6. $\displaystyle\int \frac{5x-7}{(x-1)(5x-3)}\,\mathrm{d}x$ $\qquad 2\ln|5x-3| - \ln|x-1| + C$
7. $\displaystyle\int \frac{3x-5}{x(1-x)}\,\mathrm{d}x$ $\qquad 2\ln|x-1| - 5\ln|x| + C$
8. $\displaystyle\int \frac{3x-1}{(x-2)(2x+1)}\,\mathrm{d}x$ $\qquad \frac{1}{2}\ln|2x+1| + \ln|x-2| + C$
9. $\displaystyle\int \frac{6}{x^2-2x-8}\,\mathrm{d}x$ $\qquad \ln\left|\frac{x-4}{x+2}\right| + C$
10. $\displaystyle\int \frac{x+1}{9x^2-1}\,\mathrm{d}x$ $\qquad \frac{2}{9}\ln|3x-1| - \frac{1}{9}\ln|3x+1| + C$
11. $\displaystyle\int \frac{7x-19}{x^2-2x-15}\,\mathrm{d}x$ $\qquad 5\ln|x+3| + 2\ln|x-5| + C$
12. $\displaystyle\int \frac{1}{x^2(x-1)}\,\mathrm{d}x$ $\qquad \frac{1}{x} + \ln\left|\frac{x-1}{x}\right| + C$
13. $\displaystyle\int \frac{2x^2+5x-1}{x^3+x^2-2x}\,\mathrm{d}x$ $\qquad 2\ln|x-1| + \frac{1}{2}\left|\frac{x}{x+2}\right| + C$
14. $\displaystyle\int \frac{x^2+14+1}{(x+3)(x-5)(x+7)}\,\mathrm{d}x$ $\qquad \ln\left|\frac{x^2-2x-15}{x+7}\right| + C$
15. $\displaystyle\int \frac{10x^2-23x+11}{(2-3x)(2x-1)^2}\,\mathrm{d}x$ $\qquad \frac{-2}{2x-1} - \frac{1}{3}\ln|2-3x| - \frac{1}{2}\ln|2x-1| + C$

16. $\displaystyle\int \frac{x^5+x^4+2}{x^3+x^2-x-1}\,\mathrm{d}x$

$$\begin{aligned}
\frac{x^5+x^4+2}{x^3+x^2-x-1} &= \frac{x^5+x^4-x^3+x^3-x^2+x^2-2}{x^3+x^2-x-1} \\
&= \frac{(x^5+x^4-x^3-x^2)+(x^3+x^2-2)}{x^3+x^2-x-1} \\
&= \frac{x^5+x^4-x^3-x^2}{x^3+x^2-x-1} + \frac{x^3+x^2-2}{x^3+x^2-x-1} \\
&= x^2 + \frac{x^3+x^2-x+x-1-1}{x^3+x^2-x-1} \\
&= x^2 + \frac{x^3+x^2-x-1}{x^3+x^2-x-1} + \frac{x-1}{x^3+x^2-x-1} \\
&= x^2+1+\frac{x-1}{x^3+x^2-x-1} \\
&= x^2+1+\frac{x-1}{(x+1)^2(x-1)} \\
&= x^2+1+\frac{1}{(x+1)^2} \\
\int \frac{x^5+x^4+2}{x^3+x^2-x-1}\,\mathrm{d}x &= \int \Big(x^2+1+\frac{1}{(x+1)^2}\Big)\,\mathrm{d}x \\
&= \frac{1}{3}x^3+x-\frac{1}{x+1}+C
\end{aligned}$$

2.10.7 By parts

1. $\displaystyle\int \ln x\,\mathrm{d}x$

$$\begin{matrix} u=\ln x & \Longrightarrow & \mathrm{d}u=\frac{1}{x}\mathrm{d}x \\ v=x & \Longrightarrow & \mathrm{d}v=\mathrm{d}x \end{matrix} \quad \Longrightarrow \quad \int \ln x\,\mathrm{d}x = x\ln x - \int x\frac{1}{x}\mathrm{d}x = x\ln x - x + C$$

2. $\displaystyle\int \frac{2x\sin x}{\cos^3 x}\,\mathrm{d}x$

$$\begin{aligned} u &= 2x \\ v &= \frac{1/2}{\cos^2 x} \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \mathrm{d}u &= 2\mathrm{d}x \\ \mathrm{d}v &= \frac{\sin x}{\cos^3 x}\mathrm{d}x \end{aligned}$$

Then

$$\int \frac{2x\sin x}{\cos^3 x}\,\mathrm{d}x = \frac{x}{\cos^2 x} - \int \frac{\mathrm{d}x}{\cos^2 x} = \frac{x}{\cos^2 x} - \int \sec^2 x\mathrm{d}x = \frac{x}{\cos^2 x} - \tan x + C$$

3. $\displaystyle\int x\cos x\,\mathrm{d}x$ $\qquad u=x, dv=\cos x\mathrm{d}x \Longrightarrow v=\sin x,\ \displaystyle\int x\cos x\,\mathrm{d}x = x\sin x - \int \sin x\,\mathrm{d}x = x\sin x+\cos x+C$

4. $\displaystyle\int xe^{2x}\,\mathrm{d}x$ $\qquad \dfrac{1}{2}xe^{2x}-\dfrac{1}{4}e^{2x}+C$

5. $\displaystyle\int 6xe^{3x}\,\mathrm{d}x$ $\qquad 2xe^{3x}-\dfrac{2}{3}e^{3x}+C$

6. $\displaystyle\int 25x^4\ln x\,\mathrm{d}x$ $\qquad 5x^5\ln x - x^5 + C$

7. $\displaystyle\int \cos^2 x\,\mathrm{d}x$ $\qquad \dfrac{x+\sin x\cos x}{2}+C$

8. $\displaystyle\int \sin^2 x\,\mathrm{d}x$ $\qquad \dfrac{x-\sin x\cos x}{2}+C$

9. $\displaystyle\int \frac{x^3}{(x^2+2)^2}\,\mathrm{d}x$ $\qquad \dfrac{1}{2}\ln(x^2+2)+\dfrac{1}{x^2+2}+C$

10. $\displaystyle\int e^x\sin x\,\mathrm{d}x$ $\qquad \dfrac{e^x}{2}(\sin x-\cos x)+C$

11. $\displaystyle\int x^2 2^x\,\mathrm{d}x$ $\qquad \dfrac{2^x}{\ln 2}\Big(x^2-\dfrac{2x}{\ln 2}+\dfrac{2}{\ln^2 2}\Big)+C$

2.10.8 All-together / harder

1. $\int e^{\sin x}\sin 2x \mathrm{d}x$, solve it using three techniques together: double angle formula, change of variable, and by parts

   by double angle formula $\sin 2x = 2\sin x\cos x$

$$\int e^{\sin x}\sin 2x \mathrm{d}x = 2\int e^{\sin x}\sin x\cos x \mathrm{d}x$$

   Let $t=\sin x$, then $\cos x\mathrm{d}x = \mathrm{d}t$ and thus

$$\int e^{\sin x}\sin 2x \mathrm{d}x = 2\int e^{t} t\mathrm{d}t$$

   Now we do by parts: let

$$\begin{array}{rcl} u &=& t \\ v &=& e^t \end{array} \implies \begin{array}{rcl} \mathrm{d}u &=& \mathrm{d}t \\ \mathrm{d}v &=& e^t\mathrm{d}t \end{array}$$

   Now we have

$$\int e^{\sin x}\sin 2x \mathrm{d}x = 2\int v\mathrm{d}u = 2uv - 2\int v\mathrm{d}u = 2te^t - 2\int e^t\mathrm{d}t = 2te^t - 2e^t + C$$

   Lastly

$$\int e^{\sin x}\sin 2x \mathrm{d}x = 2\sin x e^{\sin x} - 2e^{\sin x} + C$$

2. $\int_1^{3\sqrt{3}} \frac{1}{x^{2/3}(x^{2/3}+1)}\mathrm{d}x$

$$u = x^{1/3},\quad du = \frac{1}{3}x^{-2/3}\mathrm{d}x \implies \frac{1}{x^{2/3}}\mathrm{d}x = 3du, \qquad x\in[1,\ 3\sqrt{3}] \implies u\in[1,\ \sqrt{3}]$$

$$\int_1^{3\sqrt{3}} \frac{1}{x^{2/3}(x^{2/3}+1)}\mathrm{d}x \;=\; \int_1^{\sqrt{3}} \frac{3}{(u^2+1)}du \;=\; 3\tan^{-1}u\Big|_1^{\sqrt{3}} \;=\; 3\frac{\pi}{3} - 3\frac{\pi}{4} \;=\; \frac{\pi}{4}$$

3. $\int_{-5}^{5}(3+x^7\cos x)\mathrm{d}x$

$$\int_{-5}^{5}(3+x^7\cos x)\mathrm{d}x \;=\; 3\int_{-5}^{5}\mathrm{d}x + \underbrace{\int_{-5}^{5}x^7\cos x\mathrm{d}x}_{=0 \text{ odd function}} \;=\; 30$$

3 Analysis

3.1 $\epsilon - \delta$ arguments

1. **(Review the convergence of sequence)** Let $\{s_n\}$ represents a real number sequence $\{s_1, s_2, ..., s_n, ...\}$.
Now we express the definition of convergence for a sequence in $\mathbb{R}$ using logical quantifier.
Definition [Convergence of sequence] A sequence $\{s_n\}$ in $\mathbb{R}$ converges to a number $L \in \mathbb{R}$, in which we say that *the sequence* $\{s_n\}$ *converges to* L, if the following is true

$$(\forall \epsilon > 0)(\exists K \in \mathbb{N})(\forall n \in \mathbb{N})\big(n \geq K \implies |s_n - L| < \epsilon\big).$$

   (a) How do we read this logic statement? Write down the English sentence of this logic statement. How do we interpret ϵ, K and n?

   (b) Using this to prove the sequence $\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...\right\}$ converges to 0.
   - Suppose I pick $\epsilon = \frac{1}{17}$. What is K?
   - Suppose I pick $\epsilon = \frac{1}{31}$. What is K?

   (c) Using this to prove the sequence $\left\{\frac{1}{2}, \frac{2}{6}, \frac{6}{12}, ..., \frac{n!}{(n+1)!}, ...\right\}$ converges to 0.

   (d) Using this to prove the sequence $\left\{\frac{0}{2}, \frac{1}{3}, \frac{2}{3}, ..., \frac{n-1}{n+1}, ...\right\}$ converges to 1.

   (e) Using this to prove the sequence $\left\{\sqrt{n+1} - \sqrt{n}\right\}$ converges to 0. Hint $A - B = \dfrac{A^2 - B^2}{A + B}$

2. Prove that $\lim\limits_{x \to 10} (3x + 5) = 35$

3. Prove that $\lim\limits_{x \to 1} (1 - 4x) = -3$

4. Prove that $\lim\limits_{x \to -1} (x^2 + 3) = 4$

5. Prove that $\lim\limits_{x \to \infty} x = \infty$

6. $(***)$ Prove that $\lim\limits_{x \to \infty} \sqrt{x + \sqrt{x}} - \sqrt{x} = 0.5$

7. **(Extra)** Let $\{s_n\}$ represents a sequence $\{s_1, s_2, s_3, ..., s_n\}$. We consider everything in $\{s_n\}$ are real numbers in $\mathbb{R}$.
In words, a Cauchy sequence is a sequence whose elements become arbitrarily close to each other as the sequence progresses.
Now we express the definition of Cauchy sequence in $\mathbb{R}$ using logical quantifier.
Definition [Cauchy sequence] A sequence $\{s_n\}$ in $\mathbb{R}$ is a *Cauchy sequence*, if the following is true

$$(\forall \epsilon > 0)(\exists K \in \mathbb{N})(\forall n \in \mathbb{N})(\forall m \in \mathbb{N})\Big((m \geq K) \wedge (n \geq K) \implies |s_m - s_n| < \epsilon\Big).$$

8. **(Extra) Definition [Cluster Points]**

$$(\forall \epsilon > 0)(\forall m \in \mathbb{N})(\exists n \in \mathbb{N})\Big((n > m) \wedge (|s_n - L| < \epsilon)\Big).$$

9. **(Extra) Definition [Continuity]** A function f is continuous at a point c in the interval $[0, 1]$ if the following holds

$$(\forall \epsilon > 0)(\forall \delta > 0)(\forall x \in \mathbb{R})\Big(|x - c| < \delta \implies |f(x) - f(c)| < \epsilon\Big).$$

10. **(Extra) Definition [Uniform Continuity]**

$$(\forall \epsilon > 0)(\forall \delta > 0)(\forall x, y \in [0, 1])\Big(|x - y| < \delta \implies |f(x) - f(y)| < \epsilon\Big).$$

11. **(Extra) Definition [Pointwise convergence of sequences of functions]**
A sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ converges *pointwise* to the function $f : \mathbb{R} \to \mathbb{R}$

$$(\forall x \in \text{dom} f)(\forall \epsilon > 0)(\exists m \in \mathbb{N})(\forall n \in \mathbb{N})\Big(n > m \implies |f_n(x) - f(x)| < \epsilon\Big).$$

3.2 Solution to the $\epsilon - \delta$ arguments

1. **(Review the convergence of sequence)** Let $\{s_n\}$ represents a real number sequence $\{s_1, s_2, ..., s_n, ...\}$.
Now we express the definition of convergence for a sequence in $\mathbb{R}$ using logical quantifier.
Definition [Convergence of sequence] A sequence $\{s_n\}$ in $\mathbb{R}$ converges to a number $L \in \mathbb{R}$, in which we say that *the sequence* $\{s_n\}$ *converges to* L, if the following is true

$$(\forall \epsilon > 0)(\exists K \in \mathbb{N})(\forall n \in \mathbb{N})\big(n \geq K \implies |s_n - L| < \epsilon\big).$$

(a) How do we read this logic statement? Write down the English sentence of this logic statement. How do we interpret ϵ, K and n?
The *structure* of the proof that $\{s_n\}$ converges to L is like the following

"Let $\epsilon > 0$"
[A choice for K goes here]
"Let $n \in \mathbb{N}$."
"Suppose $n \geq K$."
[Proof that $|s_n - L| < \epsilon$ is true]
"Therefore, $s_n$converge to L."

(b) Using this to prove the sequence $\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...\right\}$ converges to 0.

- Suppose I pick $\epsilon = \frac{1}{17}$. What is K?
For the sequence $s_n = \frac{1}{2^{n-1}}$ and $L = 0$, we want:

$$\left|\frac{1}{2^{n-1}} - 0\right| < \frac{1}{17} \implies \frac{1}{2^{n-1}} < \frac{1}{17} \implies 2^{n-1} > 17 \implies n - 1 > \log_2(17) \approx 4.09 \implies n > 5.09$$

Since n must be an integer, we round up $n \geq 6$. Therefore, $K = 6$. This means for $\epsilon = \frac{1}{17}$, the sequence $\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\}$ will be within ϵ of 0 for all $n \geq 6$.

- Suppose I pick $\epsilon = \frac{1}{31}$. What is K?
Similar to the last question, we have

$$2^{n-1} > 31 \implies n - 1 > \log_2(31) \approx 4.95 \implies n - 1 > 4.95 \implies n > 5.95$$

Since n must be an integer, we round up we round up $n \geq 6$. Therefore, $K = 6$. This means for $\epsilon = \frac{1}{31}$, the sequence $\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\}$ will be within ϵ of 0 for all $n \geq 6$.

(c) Using this to prove the sequence $\left\{\frac{1}{2}, \frac{2}{6}, \frac{6}{12}, ..., \frac{n!}{(n+1)!}, ...\right\}$ converges to 0.
Given the sequence $s_n = \frac{n!}{(n+1)!} = \frac{n!}{(n+1)\cdot n!} = \frac{1}{n+1}$ and $L = 0$, we need to show that for any $\epsilon > 0$, there exists a $K \in \mathbb{N}$ such that for all $n \geq K$, we have: $\left|\frac{1}{n+1} - 0\right| < \epsilon$.

$$\frac{1}{n+1} < \epsilon \implies n + 1 > \frac{1}{\epsilon} \implies n > \frac{1}{\epsilon} - 1 \implies n \geq \left\lceil \frac{1}{\epsilon} - 1 \right\rceil$$

Thus, we can choose $K = \left\lceil \frac{1}{\epsilon} - 1 \right\rceil$. Then for any $\epsilon > 0$, the sequence $\left\{\frac{1}{2}, \frac{2}{6}, \frac{6}{12}, \ldots, \frac{n!}{(n+1)!}, \ldots\right\}$ will be within ϵ of 0 for all $n \geq K$.

(d) Using this to prove the sequence $\left\{\frac{0}{2}, \frac{1}{3}, \frac{2}{3}, ..., \frac{n-1}{n+1}, ...\right\}$ converges to 1.
Given the sequence $s_n = \frac{n-1}{n+1}$ and $L = 1$, we need to show that for any $\epsilon > 0$, there exists a $K \in \mathbb{N}$ such that for all $n \geq K$, we have: $\left|\frac{n-1}{n+1} - 1\right| < \epsilon$.

$$\left|\frac{n-1}{n+1} - 1\right| = \left|\frac{n - 1 - (n+1)}{n+1}\right| = \left|\frac{-2}{n+1}\right| = \frac{2}{n+1} \implies \frac{2}{n+1} < \epsilon \implies n+1 > \frac{2}{\epsilon} \implies n > \frac{2}{\epsilon} - 1 \implies n \geq \left\lceil \frac{2}{\epsilon} - 1 \right\rceil$$

Thus, we can choose $K = \left\lceil \frac{2}{\epsilon} - 1 \right\rceil$. Then for any $\epsilon > 0$, the sequence $\left\{\frac{0}{2}, \frac{1}{3}, \frac{2}{3}, \ldots, \frac{n-1}{n+1}, \ldots\right\}$ will be within ϵ of 1 for all $n \geq K$.

(e) Using this to prove the sequence $\left\{\sqrt{n+1}-\sqrt{n}\right\}$ converges to 0. Hint $A-B=\dfrac{A^2-B^2}{A+B}$

Given the sequence $s_n=\sqrt{n+1}-\sqrt{n}$, we need to show that for any $\epsilon>0$, there exists a $K\in\mathbb{N}$ such that for all $n\geq K$, we have: $\left|\sqrt{n+1}-\sqrt{n}-0\right|<\epsilon$.

Using the hint $A-B=\dfrac{A^2-B^2}{A+B}$, we rewrite $\sqrt{n+1}-\sqrt{n}$

$$\sqrt{n+1}-\sqrt{n}=\frac{(\sqrt{n+1})^2-(\sqrt{n})^2}{\sqrt{n+1}+\sqrt{n}}=\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

We need $\left|\dfrac{1}{\sqrt{n+1}+\sqrt{n}}\right|<\epsilon$. The expression is nonnegative so we can remove the absolute sign.

$$\frac{1}{\sqrt{n+1}+\sqrt{n}}<\epsilon \implies \sqrt{n+1}+\sqrt{n}>\frac{1}{\epsilon}$$

(This step is tricky) We have two term $\sqrt{n+1}>\sqrt{n}$, here we want to write ϵ as a function of n so we want to have only one term on n. We do so by the fact that $\sqrt{n+1}+\sqrt{n}$ as

$$2\sqrt{n+1}>\sqrt{n+1}+\sqrt{n}>\frac{1}{\epsilon}$$

Now we have

$$2\sqrt{n+1}>\frac{1}{\epsilon} \implies \sqrt{n+1}>\frac{1}{2\epsilon} \implies n+1>\frac{1}{4\epsilon^2} \implies n>\frac{1}{4\epsilon^2}-1 \implies n>\left\lceil\frac{1}{4\epsilon^2}-1\right\rceil$$

Thus we can choose $K=\left\lceil\dfrac{1}{4\epsilon^2}-1\right\rceil$. Then for any $\epsilon>0$, the sequence $\left\{\sqrt{n+1}-\sqrt{n}\right\}$ will be within ϵ of 0 for all $n\geq K$.

2. Prove that $\lim\limits_{x\to 10}(3x+5)=35$

Solution. Let $\epsilon>0$ be given, we determine the $\delta>0$ (which depends on ϵ) so that

$$\text{if } 0<|x-10|<\delta \text{ then } |f(x)-35|<\epsilon.$$

We begin with $|f(x)-35|<\epsilon$.

$$\begin{aligned}|f(x)-35|<\epsilon &\iff |(3x+5)-35|<\epsilon\\ &\iff |3x-30|<\epsilon\\ &\iff 3|x-10|<\epsilon\\ &\iff |x-10|<\epsilon/3\end{aligned}$$

Hence we pick $\delta=\epsilon/3$. Thus, if $0<|x-10|<\epsilon/3$, then $|f(x)-35|<\epsilon$. The proof is completed.

3. Prove that $\lim\limits_{x\to 1}(1-4x)=-3$

Solution. Let $\epsilon>0$ be given, we determine the $\delta>0$ (which depends on ϵ) so that

$$\text{if } 0<|x-1|<\delta \text{ then } |f(x)-(-3)|<\epsilon.$$

We begin with $|f(x)-(-3)|<\epsilon$.

$$\begin{aligned}|f(x)-(-3)|<\epsilon &\iff |(1-4x)+3|<\epsilon\\ &\iff |4-4x|<\epsilon\\ &\iff 4|1-x|<\epsilon\\ &\iff |1-x|<\epsilon/4\end{aligned}$$

Hence we pick $\delta=\epsilon/4$. Thus, if $0<|x-1|<\epsilon/4$, then $|f(x)-(-3)|<\epsilon$. The proof is completed.

4. Prove that $\lim\limits_{x\to -1}(x^2+3)=4$

Solution. Let $\epsilon>0$ be given, we determine the $\delta>0$ (which depends on ϵ) so that

$$\text{if } 0<|x-(-1)|<\delta \text{ then } |f(x)-4|<\epsilon. \tag{\#}$$

We begin with $|f(x)-4|<\epsilon$.

$$\begin{aligned}|f(x)-4|<\epsilon &\iff |x^2+3-4|<\epsilon\\ &\iff |x^2-1|<\epsilon\\ &\iff |(x+1)(x-1)|<\epsilon\\ &\iff |x+1||x-1|<\epsilon\end{aligned}$$

Now we need to remove the $|x-1|$, because this term does not appear in (#). Assume $\delta\leq 1$, then $x\in(-2,0)$ and $1<|x-1|<3$. Thus $|x-1||x+1|<3|x+1|<\epsilon$, which $\iff$ $|x+1|<\epsilon/3$.

Pick $\delta=\min\{1,\epsilon/3\}$. Thus, if $0<|x+1|<\min\{1,\epsilon/3\}$, then $|f(x)-4|<\epsilon$. The proof is completed.

5. Prove that $\lim\limits_{x\to\infty} x = \infty$

   This is different from the other questions because you cannot write

$$\text{if } 0 < |x-\infty| < \delta \text{ then } |x-\infty| < \epsilon \qquad \text{(this is wrong)}$$

   You cannot put ∞ inside the calculation.

   We need to show that for any given $M > 0$, there exists a $N > 0$ such that for all $x > N$, $x > M$.

   Let $M > 0$ be given. Choose $N = M$. Then, for all $x > N$, we have:

$$x > N = M$$

   Therefore, for any $M > 0$, we can find $N = M$ such that for all $x > N$, $x > M$.

6. $(***)$ Prove that $\lim\limits_{x\to\infty} \sqrt{x+\sqrt{x}} - \sqrt{x} = 0.5$

   Solution 1 Rewrite using conjugate

$$\sqrt{x+\sqrt{x}} - \sqrt{x} = \frac{(\sqrt{x+\sqrt{x}} - \sqrt{x})(\sqrt{x+\sqrt{x}} + \sqrt{x})}{\sqrt{x+\sqrt{x}} + \sqrt{x}} = \frac{x+\sqrt{x}-x}{\sqrt{x+\sqrt{x}} + \sqrt{x}} = \frac{\sqrt{x}}{\sqrt{x+\sqrt{x}} + \sqrt{x}}$$

   Simplify the denominator

$$\frac{\sqrt{x}}{\sqrt{x+\sqrt{x}} + \sqrt{x}} = \frac{\sqrt{x}}{\sqrt{x}\sqrt{1+\frac{1}{\sqrt{x}}} + \sqrt{x}} = \frac{\sqrt{x}}{\sqrt{x}\sqrt{1+\frac{1}{\sqrt{x}}} + \sqrt{x}} = \frac{1}{\sqrt{1+\frac{1}{\sqrt{x}}} + 1}$$

   We see that as $x \to \infty$, $\dfrac{1}{\sqrt{x}} \to 0$. Thus, $\sqrt{1+\dfrac{1}{\sqrt{x}}} \to \sqrt{1+0} = 1$ So, $\dfrac{1}{\sqrt{1+\frac{1}{\sqrt{x}}}+1} \to \dfrac{1}{1+1} = \dfrac{1}{2} = 0.5$

   Now we perform the $\epsilon - \delta$:

   Let $\epsilon > 0$ be given, we need to find $M > 0$ such that for all $x > M$,

$$\left| \frac{\sqrt{x}}{\sqrt{x+\sqrt{x}} + \sqrt{x}} - 0.5 \right| < \epsilon$$

   From the simplified form, we have:

$$\left| \frac{1}{\sqrt{1+\frac{1}{\sqrt{x}}} + 1} - 0.5 \right| < \epsilon$$

   Let $y = \dfrac{1}{\sqrt{x}}$. As $x \to \infty$, $y \to 0$. We need:

$$\left| \frac{1}{\sqrt{1+y}+1} - 0.5 \right| < \epsilon$$

   Simplify the expression inside the absolute value:

$$\left| \frac{1}{\sqrt{1+y}+1} - \frac{1}{2} \right| = \left| \frac{2-(\sqrt{1+y}+1)}{2(\sqrt{1+y}+1)} \right| = \left| \frac{1-\sqrt{1+y}}{2(\sqrt{1+y}+1)} \right|$$

   For y close to 0, $\sqrt{1+y} \approx 1 + \dfrac{y}{2}$. Thus,

$$\left| \frac{1-(1+\frac{y}{2})}{2(1+\frac{y}{2}+1)} \right| = \left| \frac{-\frac{y}{2}}{2(2+\frac{y}{2})} \right| = \left| \frac{-y}{4+y} \right|$$

   Since $y = \dfrac{1}{\sqrt{x}}$,

$$\left| \frac{-\frac{1}{\sqrt{x}}}{4+\frac{1}{\sqrt{x}}} \right| < \epsilon$$

   For large x, $4 + \dfrac{1}{\sqrt{x}} \approx 4$, so:

$$\left| \frac{-\frac{1}{\sqrt{x}}}{4} \right| < \epsilon \implies \frac{1}{\sqrt{x}} < 4\epsilon \implies x > \frac{1}{(4\epsilon)^2}$$

   Thus, choose $M = \dfrac{1}{(4\epsilon)^2}$. For all $x > M$,

$$\left| \sqrt{x+\sqrt{x}} - \sqrt{x} - 0.5 \right| < \epsilon$$

Solution 2 Here is a better approach: let $u = \sqrt{x}$ and hence $x = u^2$. Furthermore, for $x \to \infty$, we have $u \to \infty$.

$$\lim_{x\to\infty} \sqrt{x+\sqrt{x}} - \sqrt{x} = 0.5 \qquad \iff \qquad \lim_{u\to\infty} \sqrt{u^2+u} - u = 0.5$$

Now we do the $\epsilon - \delta$:

$$\sqrt{u^2+u} - u = \frac{(\sqrt{u^2+u}-u)(\sqrt{u^2+u}+u)}{\sqrt{u^2+u}+u} = \frac{u^2+u-u^2}{\sqrt{u^2+u}+u} = \frac{u}{\sqrt{u^2+u}+u} = \frac{u}{u\sqrt{1+\frac{1}{u}}+u} = \frac{1}{\sqrt{1+\frac{1}{u}}+1}$$

For any $\epsilon > 0$, we need to find $M > 0$ such that for all $u > M$,

$$\left| \frac{1}{\sqrt{1+\frac{1}{u}}+1} - 0.5 \right| < \epsilon$$

Let $y = \dfrac{1}{u}$. As $u \to \infty$, $y \to 0$. We need:

$$\left| \frac{1}{\sqrt{1+y}+1} - 0.5 \right| < \epsilon$$

Simplify the expression inside the absolute value:

$$\left| \frac{1}{\sqrt{1+y}+1} - \frac{1}{2} \right| = \left| \frac{2-(\sqrt{1+y}+1)}{2(\sqrt{1+y}+1)} \right| = \left| \frac{1-\sqrt{1+y}}{2(\sqrt{1+y}+1)} \right|$$

For y close to 0, $\sqrt{1+y} \approx 1 + \dfrac{y}{2}$. Thus,

$$\left| \frac{1-(1+\frac{y}{2})}{2(1+\frac{y}{2}+1)} \right| = \left| \frac{-\frac{y}{2}}{2(2+\frac{y}{2})} \right| = \left| \frac{-y}{4+y} \right|$$

Since $y = \dfrac{1}{u}$,

$$\left| \frac{-\frac{1}{u}}{4+\frac{1}{u}} \right| < \epsilon$$

For large u, $4 + \dfrac{1}{u} \approx 4$, so:

$$\left| \frac{-\frac{1}{u}}{4} \right| < \epsilon \implies \frac{1}{u} < 4\epsilon \implies u > \frac{1}{(4\epsilon)^2}$$

Thus, choose $M = \dfrac{1}{(4\epsilon)^2}$. For all $u > M$,

$$\left| \sqrt{u^2+u} - u - 0.5 \right| < \epsilon$$

3.3 Limit: true or false

1. True or false: If $a_n \to 0$ and b_n is bounded, then $a_n b_n \to 0$
2. True or false: If a_n^2 is convergent, then a_n is convergent
3. True or false: If a_n and b_n are divergent, then $a_n + b_n$ is divergent
4. True or false: If a_n is convergent and $b_n \to \infty$, then $\dfrac{a_n}{b_n} \to 0$
5. True or false: If a_n is bounded and $b_n \to \infty$, then $\dfrac{a_n}{b_n} \to 0$
6. True or false: If a_n is bounded and b_n is a sequence that $b_n = a_n$ for any $n > 100$, then b_n is bounded.
7. True or false: If $a_n \to a$ and b_n is a sequence that $b_n = a_n$ for any $n > 100$, then $b_n \to a$ is bounded.
8. True or false: If a_n and b_n are convergent and $a_n < b_n$ for all n, then $\lim\limits_{n\to\infty} a_n < \lim\limits_{n\to\infty} b_n$
9. True or false: If a_n and b_n are convergent and $a_n \leq b_n$ for all n, then $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$
10. True or false: If $\lim\limits_{n\to\infty} a_n = a$, then $\lim\limits_{n\to\infty} a_{n+1} = a$
11. True or false: If $\lim\limits_{n\to\infty} a_n = a$, then $\lim\limits_{n\to\infty} a_{2n} = a$
12. True or false: If a_n is convergent, then $\lim\limits_{n\to\infty} (a_{n+1} - a_n) = 0$
13. True or false: If $\lim\limits_{n\to\infty} (a_{n+1} - a_n) = 0$, then a_n is convergent

3.4 Limit: true or false solution

1. True or false: If $a_n \to 0$ and b_n is bounded, then $a_n b_n \to 0$ — true
2. True or false: If a_n^2 is convergent, then a_n is convergent — false, counter example $a_n = (-1)^n$
3. True or false: If a_n and b_n are divergent, then $a_n + b_n$ is divergent — false, counter example $a_n = n, b_n = -n$
4. True or false: If a_n is convergent and $b_n \to \infty$, then $\dfrac{a_n}{b_n} \to 0$ — true
5. True or false: If a_n is bounded and $b_n \to \infty$, then $\dfrac{a_n}{b_n} \to 0$ — true
6. True or false: If a_n is bounded and b_n is a sequence that $b_n = a_n$ for any $n > 100$, then b_n is bounded. — true
7. True or false: If $a_n \to a$ and b_n is a sequence that $b_n = a_n$ for any $n > 100$, then $b_n \to a$ is bounded. — true
8. True or false: If a_n and b_n are convergent and $a_n < b_n$ for all n, then $\lim\limits_{n\to\infty} a_n < \lim\limits_{n\to\infty} b_n$ — false, $a_n = 0, b_n = \dfrac{1}{n}$
9. True or false: If a_n and b_n are convergent and $a_n \leq b_n$ for all n, then $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$ — true
10. True or false: If $\lim\limits_{n\to\infty} a_n = a$, then $\lim\limits_{n\to\infty} a_{n+1} = a$ — true
11. True or false: If $\lim\limits_{n\to\infty} a_n = a$, then $\lim\limits_{n\to\infty} a_{2n} = a$ — true
12. True or false: If a_n is convergent, then $\lim\limits_{n\to\infty} (a_{n+1} - a_n) = 0$ — true
13. True or false: If $\lim\limits_{n\to\infty} (a_{n+1} - a_n) = 0$, then a_n is convergent — false, $a_n = \sqrt{n}$

$$
\begin{aligned}
\lim_{n\to\infty} (a_{n+1} - a_n) &= \lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} \\
&= \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\
&= \lim_{n\to\infty} \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\infty + \infty} = 0
\end{aligned}
$$

But $\sqrt{n} \to \infty$ is divergent

3.5 Squeeze Theorem

1. If $6 - x^2 \leq f(x) \leq 6 + x^2$ for all x, then what is $\lim_{x \to 0} f(x)$?
2. Find $\lim_{n \to \infty} \frac{\sin(n)}{n}$
3. Find $\lim_{n \to \infty} \frac{\sin(n)}{n^2}$
4. Find $\lim_{n \to \infty} 4 + \frac{\cos(n)}{\sqrt{n}}$
5. Find $\lim_{n \to \infty} \frac{(-1)^n}{\sqrt{n}} - 1$
6. Explain why you cannot use squeeze theorem to find $\lim_{n \to \infty} (-1)^n$
7. Find $\lim_{x \to 0} x \sin x$
8. Find $\lim_{x \to 0^-} x^3 \cos \frac{3}{x}$

3.6 Squeeze Theorem solution

1. If $6 - x^2 \leq f(x) \leq 6 + x^2$ for all x, then what is $\lim_{x\to 0} f(x)$?

$$6$$

2. Find $\lim_{n\to\infty} \dfrac{\sin(n)}{n}$

$$\frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n} \quad \forall n \quad \Longrightarrow \lim_{n\to\infty} \frac{\sin(n)}{n} = 0$$

3. Find $\lim_{n\to\infty} \dfrac{\sin(n)}{n^2}$

$$\frac{-1}{n^2} \leq \frac{\sin(n)}{n^2} \leq \frac{1}{n^2} \quad \forall n \quad \Longrightarrow \lim_{n\to\infty} \frac{\sin(n)}{n^2} = 0$$

4. Find $\lim_{n\to\infty} 4 + \dfrac{\cos(n)}{\sqrt{n}}$

$$4 + \frac{-1}{\sqrt{n}} \leq 4 + \frac{\cos(n)}{\sqrt{n}} \leq 4 + \frac{1}{\sqrt{n}} \quad \forall n \quad \Longrightarrow \lim_{n\to\infty} 4 + \frac{\cos(n)}{\sqrt{n}} = 4$$

5. Find $\lim_{n\to\infty} \dfrac{(-1)^n}{\sqrt{n}} - 1$

$$\frac{-1}{\sqrt{n}} - 1 \leq \frac{(-1)^n}{\sqrt{n}} - 1 \leq \frac{1}{\sqrt{n}} - 1 \quad \forall n \quad \Longrightarrow \lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n}} - 1 = -1$$

6. Explain why you cannot use squeeze theorem to find $\lim_{n\to\infty} (-1)^n$
The setting of squeeze theorem is that if sequences $a_n \leq b_n \leq c_n$ for all n, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then b_n is convergent and $\lim_{n\to\infty} b_n = L$. The theorem requires $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, which is not the case for $\lim_{n\to\infty} (-1)^n$.

7. Find $\lim_{x\to 0} x \sin x$

$$\lim_{x\to 0} x \sin x = \lim_{x\to 0} \frac{\sin x}{\frac{1}{x}}$$

Let $y = \dfrac{1}{x}$ then $x = \dfrac{1}{y}$, and $x \to 0$ leads to $y \to \infty$

$$\lim_{x\to 0} \frac{\sin x}{\frac{1}{x}} = \lim_{y\to\infty} \frac{\sin \frac{1}{y}}{y}$$

Now

$$\frac{-1}{y} \leq \frac{\sin \frac{1}{y}}{y} \leq \frac{1}{y} \qquad \Longrightarrow \qquad \lim_{y\to\infty} \frac{\sin \frac{1}{y}}{y} = 0 \Longrightarrow \lim_{x\to 0} x \sin x = 0$$

8. Find $\lim_{x\to 0^-} x^3 \cos \dfrac{3}{x}$

$$\lim_{x\to 0^-} x^3 \cos \frac{3}{x} = \lim_{x\to 0^-} \frac{\cos \frac{3}{x}}{\frac{1}{x^3}}$$

Let $y = \dfrac{1}{x^3}$ then $\dfrac{1}{x} = y^{\frac{1}{3}}$, and $x \to 0^-$ leads to $y \to -\infty$

$$\lim_{x\to 0^-} \frac{\cos \frac{3}{x}}{\frac{1}{x^3}} = \lim_{y\to -\infty} \frac{\cos 3y^{\frac{1}{3}}}{y}$$

Now

$$\frac{-1}{y} \leq \frac{\cos 3y^{\frac{1}{3}}}{y} \leq \frac{1}{y} \qquad \Longrightarrow \qquad \lim_{y\to -\infty} \frac{\cos 3y^{\frac{1}{3}}}{y} = 0 \Longrightarrow \lim_{x\to 0^-} x^3 \cos \frac{3}{x} = 0$$

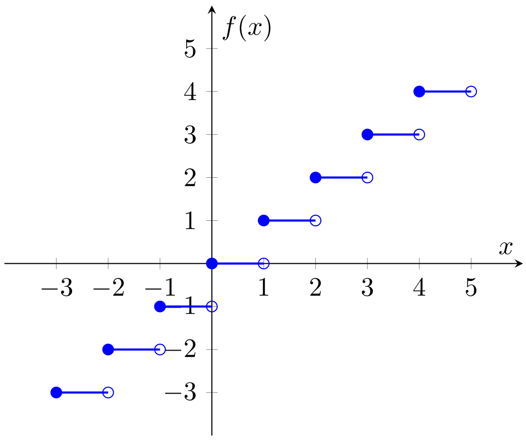
3.7 Differentiability

1. Let $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \begin{cases} \dfrac{x}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Find $f'(x)$ for $x \neq 0$. Is f differentiable at $x = 0$?

2. Let $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \begin{cases} x \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Find $f'(x)$ for $x \neq 0$. Is f differentiable at $x = 0$?

3. Let $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Find $f'(x)$ for $x \neq 0$. Is f differentiable at $x = 0$?

4. Let $f(x) = \lceil x \rceil := \min\{n \in \mathbb{Z} \mid n \geq x\}$



Find $f'(1.5)$. Is f differentiable at $x = 1$?

3.8 Differentiability solution

1. Let $f:\mathbb{R}\to\mathbb{R}$ defined as $f(x)=\begin{cases}\dfrac{x}{|x|} & x\neq 0\\ 0 & x=0\end{cases}$. Find $f'(x)$ for $x\neq 0$. Is f differentiable at $x=0$?

For $x\neq 0$, we have

$$f(x)=\begin{cases}x & x>0\\ -x & x<0\end{cases}$$

Therefore

$$f'(x)=\begin{cases}1 & x>0\\ -1 & x<0\end{cases}$$

To determine is f differentiable at $x=0$, we go back to the (left/right) limit definition

$$f'(0)\text{ exists}\iff \lim_{h\to 0^-}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^+}\frac{f(h)-f(0)}{h}$$

Since $f(x)=0$ for $x=0$, so we need to check

$$\lim_{h\to 0^-}\frac{f(h)}{h}\stackrel{?}{=}\lim_{h\to 0^+}\frac{f(h)}{h}\iff \lim_{h\to 0^-}\frac{-1}{h}\stackrel{?}{=}\lim_{h\to 0^+}\frac{+1}{h}$$

Since both limit converges to infinity

$$\lim_{h\to 0^-}\frac{-1}{h}=-\infty\qquad \lim_{h\to 0^+}\frac{+1}{h}=\infty$$

so they both does not exist and therefore f is not differentiable at $x=0$.

Extra explanation You actually do not need to compute both the limits. Once you see one of them does not exist, then you can stop, because "does not exist" means the solution is $\varnothing$, and no number in $\mathbb{R}$ will equal to $\varnothing$, so we can immediately end the work and say f is not differentiable

2. Let $f:\mathbb{R}\to\mathbb{R}$ defined as $f(x)=\begin{cases}x\sin\dfrac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}$. Find $f'(x)$ for $x\neq 0$. Is f differentiable at $x=0$?

For $x\neq 0$, we have $f(x)=x\sin\dfrac{1}{x}$. To find $f'(x)$, we use the product rule: $f'(x)=\sin\dfrac{1}{x}-x\dfrac{1}{x^2}\cos\dfrac{1}{x}=\sin\dfrac{1}{x}-\dfrac{1}{x}\cos\dfrac{1}{x}$

To determine is f differentiable at $x=0$, we go back to the limit definition

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

Since $f(0)=0$ by the definition of f, we have

$$f'(0)=\lim_{h\to 0}\frac{h\sin\frac{1}{h}}{h}=\lim_{h\to 0}\sin\frac{1}{h}$$

This limit does not exist so f is not differentiable at $x=0$.

Explanation why $\lim\limits_{h\to 0}\sin\dfrac{1}{h}$ does not exist: consider change of variable, let $k=\dfrac{1}{h}$, so now for $h\to 0$ we have $k\to\infty$, the limit now becomes

$$f'(0)=\lim_{h\to 0}\sin\frac{1}{h}=\lim_{k\to\infty}\sin k$$

Since we know $\sin$ is oscillating so it does not converge if k goes to infinity.

3. Let $f:\mathbb{R}\to\mathbb{R}$ defined as $f(x)=\begin{cases}x^2\sin\dfrac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}$. Find $f'(x)$ for $x\neq 0$. Is f differentiable at $x=0$?

For $x\neq 0$, we have $f(x)=x^2\sin\dfrac{1}{x}$. To find $f'(x)$, we use the product rule: $f'(x)=2x\sin\dfrac{1}{x}-x^2\dfrac{1}{x^2}\cos\dfrac{1}{x}=2x\sin\dfrac{1}{x}-\cos\dfrac{1}{x}$

To determine is f differentiable at $x=0$, we go back to the limit definition

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

Since $f(0)=0$ by the definition of f, we have

$$f'(0)=\lim_{h\to 0}\frac{h^2\sin\frac{1}{h}}{h}=\lim_{h\to 0}h\sin\frac{1}{h}$$

Now we get stuck at this limit. Consider change of variable, let $k = \dfrac{1}{h}$, so now for $h \to 0$ we have $k \to \infty$, the limit now becomes

$$f'(0) = \lim_{h\to 0} h \sin\frac{1}{h} = \lim_{k\to\infty} \frac{\sin k}{k}$$

Now since $-1 < \sin k < +1$, thus

$$\underbrace{\lim_{k\to\infty} \frac{-1}{k}}_{=0} \leq \lim_{k\to\infty} \frac{\sin k}{k} \leq \underbrace{\lim_{k\to\infty} \frac{1}{k}}_{=0}$$

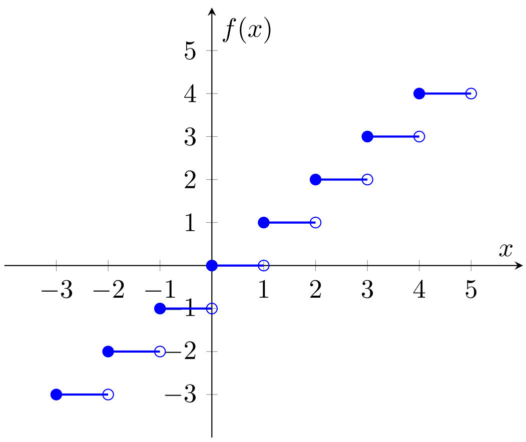
by squeeze theorem the limit is zero.

Extra content ($f'(x)$ is not continuous at $x = 0$)

$$f'(x) = \begin{cases} 2x \sin\dfrac{1}{x} - \cos\dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Note that $\lim_{x\to 0} f'(x) = \lim_{x\to 0} \left(2x\sin\dfrac{1}{x} - \cos\dfrac{1}{x}\right)$ does not exist (it is at ∞), so $f'(x)$ is not continuous at $x = 0$

4. Let $f(x) = \lceil x \rceil := \min\{n \in \mathbb{Z} \mid n \geq x\}$



Find $f'(1.5)$. Is f differentiable at $x = 1$?

$$f'(1.5) = \frac{\mathrm{d}}{\mathrm{d}x} f(1.5) = \frac{\mathrm{d}}{\mathrm{d}x} 1 = 0$$

(From the figure, we see that the slop of f at 1.5 is a horizontal line, so the slope $= 0$)

$$\begin{aligned} \lim_{h\to 0-} \frac{\lceil 1+h \rceil - \lceil 1 \rceil}{h} &= \lim_{h\to 0-} \frac{1-1}{h} = \lim_{h\to 0-} \frac{0}{h} = \lim_{h\to 0-} 0 = 0 \\ \lim_{h\to 0+} \frac{\lceil 1+h \rceil - \lceil 1 \rceil}{h} &= \lim_{h\to 0+} \frac{2-1}{h} = \lim_{h\to 0+} \frac{1}{h} = +\infty \end{aligned}$$

Hence $\lim_{h\to 0-} \dfrac{\lceil 1+h \rceil - \lceil 1 \rceil}{h} \neq \lim_{h\to 0+} \dfrac{\lceil 1+h \rceil - \lceil 1 \rceil}{h}$ so f is not differentiable at $x = 1$

3.9 Analysis of integral

1. Given $f(x)$ is an even function. Prove that $F(x) = \displaystyle\int_0^x f(t)\mathrm{d}t$ is an odd function.

2. Let $f(x) = x$. Using Riemann sum, prove that $\displaystyle\int_0^1 f(x)\mathrm{d}x = \frac{1}{2}$.

3. Let $f(x) = 10$. Using Riemann sum, prove that $\displaystyle\int_3^5 f(x)\mathrm{d}x = 20$.

4. Let $f(x) = e^x$. Using Riemann sum, prove that $\displaystyle\int_0^1 f(x)\mathrm{d}x = e - 1$.

3.10 Analysis of integral solution

1. Given $f(x)$ is an even function. Prove that $F(x) = \displaystyle\int_0^x f(t)\mathrm{d}t$ is an odd function.

$$F(-x) = \int_0^{-x} f(t)\mathrm{d}t = \underbrace{\int_0^x f(-u)(-\mathrm{d}u)}_{u=-t} = -\int_0^x f(u)\mathrm{d}u = -F(x)$$

2. Let $f(x) = x$. Using Riemann sum, prove that $\displaystyle\int_0^1 f(x)\mathrm{d}x = \frac{1}{2}$.

   First write down the summation

$$\int_0^1 f(x)\mathrm{d}x \;=\; \lim_{N\to\infty}\sum_{k=0}^{N} f(a+kh)h.$$

   The meaning is, we chop the integration range into N interval with width h, where

$$h = \frac{b-a}{N}$$

   with $b = 1$ and $a = 0$. Now we have

$$\int_0^1 f(x)\mathrm{d}x \;=\; \lim_{N\to\infty}\sum_{k=0}^{N} f\Big(0 + k\frac{1-0}{N}\Big)\frac{1-0}{N} \;=\; \lim_{N\to\infty}\sum_{k=0}^{N} f\Big(\frac{k}{N}\Big)\frac{1}{N}$$

   Now for $f(x) = x$, we have $f\Big(\dfrac{k}{N}\Big) = \dfrac{k}{N}$, therefore

$$\int_0^1 f(x)\mathrm{d}x \;=\; \lim_{N\to\infty}\sum_{k=0}^{N}\frac{k}{N}\frac{1}{N} \;=\; \lim_{N\to\infty}\frac{1}{N^2}\sum_{k=0}^{N} k$$

   Now we "kill" the summation sign,

$$\int_0^1 f(x)\mathrm{d}x \;=\; \lim_{N\to\infty}\frac{1}{N^2}\frac{N(N+1)}{2} \;=\; \lim_{N\to\infty}\frac{N+1}{2N} \;=\; \lim_{N\to\infty}\frac{1+1/N}{2} \;=\; \frac{1}{2}$$

3. Let $f(x) = 10$. Using Riemann sum, prove that $\displaystyle\int_3^5 f(x)\mathrm{d}x = 20$.

   First write down the summation

$$\int_0^1 f(x)\mathrm{d}x \;=\; \lim_{N\to\infty}\sum_{k=0}^{N-1} f(a+kh)h.$$

   The meaning is, we chop the integration range into N interval with width h, where

$$h = \frac{b-a}{N}$$

   with $b = 5$ and $a = 3$. Now we have

$$\int_0^1 f(x)\mathrm{d}x \;=\; \lim_{N\to\infty}\sum_{k=0}^{N-1} f\Big(0 + k\frac{5-3}{N}\Big)\frac{5-3}{N} \;=\; 2\lim_{N\to\infty}\sum_{k=0}^{N-1} f\Big(\frac{2k}{N}\Big)\frac{1}{N}$$

   Now for $f(x) = 10$, we have $f\Big(\dfrac{2k}{N}\Big) = 10$, therefore

$$\int_0^1 f(x)\mathrm{d}x \;=\; 2\lim_{N\to\infty}\sum_{k=0}^{N-1} 10\frac{1}{N} \;=\; \lim_{N\to\infty}\frac{20}{N}\sum_{k=0}^{N-1} 1$$

   Now we "kill" the summation sign,

$$\int_0^1 f(x)\mathrm{d}x \;=\; \lim_{N\to\infty}\frac{20}{N}N \;=\; \lim_{N\to\infty} 20 \;=\; 20$$

4. Let $f(x) = e^x$. Using Riemann sum, prove that $\displaystyle\int_0^1 f(x)\mathrm{d}x = e - 1$.

   $a = 0, b = 1, h = \dfrac{b-a}{n} = \dfrac{1-0}{n}$, and $f(a+kh) = f(0+kh) = f(kh) = \exp\Big(\dfrac{k}{n}\Big)$

$$\int_0^1 e^x\mathrm{d}x = \lim_{n\to\infty}\sum_{k=0}^{n-1}\exp\Big(\frac{k}{n}\Big)\cdot\frac{1-0}{n} = \lim_{n\to\infty}\frac{1+\exp\left(\frac{1}{n}\right)+\cdots+\exp\left(\frac{n-1}{n}\right)}{n}$$

By $1 + r + r^2 + \cdots + r^{n-1} = \dfrac{1-r^n}{1-r}$, we have

$$= \lim_{n\to\infty} \frac{1-e^{n/n}}{(1-e^{1/n})n} = \lim_{n\to\infty} \frac{1-e}{(1-e^{1/n})n} = \lim_{n\to\infty} \frac{e-1}{(e^{1/n}-1)n} = (e-1)\lim_{n\to\infty} \frac{1}{(e^{1/n}-1)n}$$

So if we show $\lim\limits_{n\to\infty} \dfrac{1}{(e^{1/n}-1)n} = 1$ then we finish.

$$\lim_{n\to\infty} \frac{1}{(e^{1/n}-1)n} = \lim_{n\to\infty} \frac{1/n}{e^{1/n}-1} = \lim_{z\to 0} \frac{z}{e^z-1} \overset{LH}{=} \lim_{z\to 0} \frac{1}{e^z} = \frac{1}{1} = 1$$

4 Computation in Linear algebra

4.1 Basic computation

1. Given
$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 6 & -2 \end{bmatrix},\ \boldsymbol{B} = \begin{bmatrix} 1 & 6 \\ 3 & -2 \end{bmatrix},\ \boldsymbol{C} = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix},\ \boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \boldsymbol{u} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
   (a) Find $\boldsymbol{u} - 2\boldsymbol{v}$
   (b) Find $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}$
   (c) Find $\boldsymbol{u}^\top \boldsymbol{w}$ and $\boldsymbol{w}^\top \boldsymbol{v}$
   (d) Find $\boldsymbol{A}\boldsymbol{v}$
   (e) Find $\boldsymbol{B}\boldsymbol{w} - \boldsymbol{A}\boldsymbol{u}$
   (f) Find $(\boldsymbol{A} + \boldsymbol{I})\boldsymbol{u}$
   (g) Find $\langle \boldsymbol{C}\boldsymbol{A}\boldsymbol{u}, \boldsymbol{w} \rangle$
   (h) Find $\boldsymbol{C}\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{C}$
   (i) Find $\boldsymbol{B}^2$
   (j) Find $\boldsymbol{B}^2 - \boldsymbol{A}\boldsymbol{I}$

2. True or false: for any three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$, we have $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$

3. Compute
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \qquad \begin{pmatrix} \sqrt{2}-1 & 2 \\ 2 & \sqrt{2}+1 \end{pmatrix} \begin{bmatrix} \sqrt{2}+1 \\ \sqrt{2}-1 \end{bmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix} \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}$$

4. Given $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, find $\boldsymbol{A} + \boldsymbol{B}$, $10\boldsymbol{A} - \boldsymbol{B}$, $\boldsymbol{A}\boldsymbol{B}$ and $\boldsymbol{B}\boldsymbol{A}$

5. Given $\boldsymbol{A} = \begin{bmatrix} 6 & -3 \\ 3 & 6 \end{bmatrix}$, find $\boldsymbol{A}\boldsymbol{A}^\top - 45\boldsymbol{I}$

6. Given $\boldsymbol{A} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$, find $\boldsymbol{A}\boldsymbol{B}$ and $\boldsymbol{B}\boldsymbol{A}$

7. Given $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -2 \\ 3 & 0 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 3 & -2 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix}$, find $2\boldsymbol{A}\boldsymbol{B}$

8. Given $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 4 & 2 & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$, verify $(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{C} = \boldsymbol{A}\boldsymbol{C} + \boldsymbol{B}\boldsymbol{C}$ and $(\boldsymbol{A}\boldsymbol{B})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{B}\boldsymbol{C})$

9. Given $\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix},\ \boldsymbol{y} = \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix},\ \boldsymbol{z} = \begin{bmatrix} 7 & -8 & -9 \end{bmatrix}$. Find $\boldsymbol{x}^\top \boldsymbol{y}$, $\langle \boldsymbol{y}, \boldsymbol{z}^\top \rangle$ and $\boldsymbol{y}\boldsymbol{z}$. Does $\langle \boldsymbol{y}, \boldsymbol{z}^\top \rangle = \boldsymbol{y}\boldsymbol{z}$?

10. Compute $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 4 & -1 & 2 \\ -1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

11. Write $3x^2 - 8xy + 2y^2 + 6xz - 3z^2$ in matrix form as in the previous question.

12. $\boldsymbol{A} = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix},\ \boldsymbol{B} = \begin{bmatrix} 2 & 3 & 1 \\ 2 & -2 & -2 \\ -1 & 2 & 1 \end{bmatrix}\ \boldsymbol{C} = \begin{bmatrix} 3 & 1 & -3 \\ 0 & 2 & 6 \\ -1 & 2 & 1 \end{bmatrix}$. Decide whether $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{A}\boldsymbol{C}$.

13. Given $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, find $\|\boldsymbol{x}\|$, $\|\boldsymbol{x}\|^2$ and the angle between $\boldsymbol{x}, \boldsymbol{y}$

14. Given $\boldsymbol{u} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$, find $\boldsymbol{u}^\top \boldsymbol{v}$, $\|\boldsymbol{u}\|$, $\|\boldsymbol{v}\|$ and the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$

15. Given $\boldsymbol{u} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$, is $|\boldsymbol{u}^\top \boldsymbol{v}| \leq \|\boldsymbol{u}\| \|\boldsymbol{v}\|$ true?

16. Given $\boldsymbol{u} = \begin{bmatrix} 1 \\ -3 \\ 4 \\ 2 \end{bmatrix}$, find $\dfrac{\boldsymbol{u}}{\|\boldsymbol{u}\|}$

17. What is the Euclidean distance between $\boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$?

18. Find t such that if $\boldsymbol{A} = \begin{bmatrix} 1 & t \\ 3 & 1 \end{bmatrix}$, we have $\boldsymbol{A}^\top = \boldsymbol{A}$

19. Given $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 0 \\ -3 & -3 & -3 \\ 0 & 0 & 4 \end{bmatrix}$, find $\boldsymbol{A}^\top$, $\boldsymbol{A}\boldsymbol{A}^\top$ and $\boldsymbol{A}^\top\boldsymbol{A}$, is $\boldsymbol{A}\boldsymbol{A}^\top = \boldsymbol{A}^\top\boldsymbol{A}$?

20. Given $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, find $(\boldsymbol{A}\boldsymbol{B})^\top$, $\boldsymbol{B}^\top\boldsymbol{A}^\top$, is $(\boldsymbol{A}\boldsymbol{B})^\top = \boldsymbol{B}^\top\boldsymbol{A}^\top$?

21. A matrix $\boldsymbol{A}$ is called orthogonal if $\boldsymbol{A}\boldsymbol{A}^\top = \boldsymbol{I}$. Which of the following matrices is orthogonal?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}$$

22. Given $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Find $\boldsymbol{A}^2, \boldsymbol{A}^3, \boldsymbol{A}^4$, do you find something?

23. True or false: the diagonal of a matrix $\boldsymbol{A}$ and the diagonal of a matrix $A^\top$ are always the same for any square matrix $\boldsymbol{A}$

24. $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 3 \end{bmatrix}$, turn $\boldsymbol{A}$ into a row echelon form.

25. $\boldsymbol{A} = \begin{bmatrix} -4 & 1 & -6 \\ 1 & 2 & -5 \\ 6 & 3 & -4 \end{bmatrix}$, turn $\boldsymbol{A}$ into a row echelon form.

26. $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -4 & -4 \\ 2 & 5 & -9 & -10 \\ 3 & -2 & 3 & 11 \end{bmatrix}$, turn $\boldsymbol{A}$ into a row echelon form.

4.2 Basic computation solution

1. Given
$$\boldsymbol{A}=\begin{bmatrix}1 & 3\\ 6 & -2\end{bmatrix},\ \boldsymbol{B}=\begin{bmatrix}1 & 6\\ 3 & -2\end{bmatrix},\ \boldsymbol{C}=\begin{bmatrix}0 & -1\\ -1 & -1\end{bmatrix},\ \boldsymbol{I}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\boldsymbol{u}=\begin{bmatrix}0\\ -1\end{bmatrix},\boldsymbol{v}=\begin{bmatrix}1\\ 2\end{bmatrix},\boldsymbol{w}=\begin{bmatrix}2\\ 1\end{bmatrix}$$

   (a) Find $\boldsymbol{u}-2\boldsymbol{v}$ $\qquad\begin{bmatrix}-2\\ -5\end{bmatrix}$

   (b) Find $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}$ $\qquad\begin{bmatrix}3\\ 2\end{bmatrix}$

   (c) Find $\boldsymbol{u}^\top\boldsymbol{w}$ and $\boldsymbol{w}^\top\boldsymbol{v}$ $\qquad -1$ and 4

   (d) Find $\boldsymbol{A}\boldsymbol{v}$ $\qquad 1\begin{bmatrix}1\\ 6\end{bmatrix}+2\begin{bmatrix}3\\ -2\end{bmatrix}=\begin{bmatrix}7\\ 2\end{bmatrix}$

   (e) Find $\boldsymbol{B}\boldsymbol{w}-\boldsymbol{A}\boldsymbol{u}$ $\qquad\begin{bmatrix}11\\ 2\end{bmatrix}$

   (f) Find $(\boldsymbol{A}+\boldsymbol{I})\boldsymbol{u}$ $\qquad$ same as $\boldsymbol{A}\boldsymbol{u}+\boldsymbol{u}=\begin{bmatrix}-3\\ 2\end{bmatrix}+\begin{bmatrix}0\\ -1\end{bmatrix}=\begin{bmatrix}-3\\ 1\end{bmatrix}$

   (g) Find $\langle\boldsymbol{C}\boldsymbol{A}\boldsymbol{u},\boldsymbol{w}\rangle$ $\qquad \boldsymbol{C}\boldsymbol{A}\boldsymbol{u}=\begin{bmatrix}-2\\ 1\end{bmatrix}$, then $(\boldsymbol{C}\boldsymbol{A}\boldsymbol{u})^\top\boldsymbol{w}=-3$

   (h) Find $\boldsymbol{C}\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{C}$ $\qquad \boldsymbol{C}\boldsymbol{A}=\begin{bmatrix}-6 & 2\\ -7 & -1\end{bmatrix},\ \boldsymbol{A}\boldsymbol{C}=\begin{bmatrix}-3 & -4\\ 2 & -4\end{bmatrix}$

   (i) Find $\boldsymbol{B}^2$ $\qquad\begin{bmatrix}19 & -6\\ -322 & \end{bmatrix}$

   (j) Find $\boldsymbol{B}^2-\boldsymbol{A}\boldsymbol{I}\boldsymbol{C}$ $\qquad \boldsymbol{B}^2-\boldsymbol{A}\boldsymbol{I}\boldsymbol{C}=\boldsymbol{B}^2-\boldsymbol{A}\boldsymbol{C}=\begin{bmatrix}22 & -2\\ -5 & 26\end{bmatrix}$

2. True or false: for any three vectors $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$, we have $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$ $\qquad$ true

3. Compute
$$\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{bmatrix}5\\ 6\end{bmatrix}=\begin{bmatrix}5+12\\ 15+24\end{bmatrix}=\begin{bmatrix}17\\ 39\end{bmatrix}$$
$$\begin{pmatrix}\sqrt{2}-1 & 2\\ 2 & \sqrt{2}+1\end{pmatrix}\begin{bmatrix}\sqrt{2}+1\\ \sqrt{2}-1\end{bmatrix}=\begin{bmatrix}2\sqrt{2}-1\\ 2\sqrt{2}+3\end{bmatrix}\qquad\begin{pmatrix}1 & 0 & 0 & 0\\ 2 & 1 & 0 & 0\\ 0 & 2 & 1 & 0\\ 0 & 0 & 2 & 1\end{pmatrix}\begin{bmatrix}10\\ 10\\ 10\\ 10\end{bmatrix}=\begin{bmatrix}10\\ 30\\ 30\\ 30\end{bmatrix}$$

4. Given $\boldsymbol{A}=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix},\boldsymbol{B}=\begin{bmatrix}5 & 6\\ 7 & 8\end{bmatrix}$, find $\boldsymbol{A}+\boldsymbol{B}$, $10\boldsymbol{A}-\boldsymbol{B}$, $\boldsymbol{A}\boldsymbol{B}$ and $\boldsymbol{B}\boldsymbol{A}$
$$\boldsymbol{A}+\boldsymbol{B}=\begin{bmatrix}6 & 8\\ 10 & 12\end{bmatrix},\quad 10\boldsymbol{A}-\boldsymbol{B}=\begin{bmatrix}5 & 14\\ 23 & 32\end{bmatrix},\quad \boldsymbol{A}\boldsymbol{B}=\begin{bmatrix}19 & 22\\ 43 & 50\end{bmatrix},\quad \boldsymbol{B}\boldsymbol{A}=\begin{bmatrix}23 & 34\\ 31 & 46\end{bmatrix}$$

5. Given $\boldsymbol{A}=\begin{bmatrix}6 & -3\\ 3 & 6\end{bmatrix}$, find $\boldsymbol{A}\boldsymbol{A}^\top-45\boldsymbol{I}$
$$\boldsymbol{A}\boldsymbol{A}^\top-45\boldsymbol{I}=\begin{bmatrix}6 & -3\\ 3 & 6\end{bmatrix}\begin{bmatrix}6 & 3\\ -3 & 6\end{bmatrix}-\begin{bmatrix}45 & 0\\ 0 & 45\end{bmatrix}=\begin{bmatrix}45 & 0\\ 0 & 45\end{bmatrix}-\begin{bmatrix}45 & 0\\ 0 & 45\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

6. Given $\boldsymbol{A}=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix},\boldsymbol{B}=\begin{bmatrix}4 & 5 & 6\end{bmatrix}$, find $\boldsymbol{A}\boldsymbol{B}$ and $\boldsymbol{B}\boldsymbol{A}$
$$\boldsymbol{A}\boldsymbol{B}=\begin{bmatrix}4 & 5 & 6\\ 8 & 10 & 12\\ 12 & 15 & 18\end{bmatrix}\qquad \boldsymbol{B}\boldsymbol{A}=32$$

7. Given $\boldsymbol{A}=\begin{bmatrix}1 & 2 & -2\\ 3 & 0 & 1\end{bmatrix},\boldsymbol{B}=\begin{bmatrix}2 & 1 & 0 & -1\\ 1 & 3 & -2 & 0\\ 0 & 1 & -1 & -1\end{bmatrix}$, find $2\boldsymbol{A}\boldsymbol{B}$
$$2\boldsymbol{A}\boldsymbol{B}=\begin{bmatrix}8 & 10 & -4 & 2\\ 12 & 8 & -2 & -8\end{bmatrix}$$

8. Given $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 4 & 2 & 0 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$, $\boldsymbol{C} = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$, verify $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC}+\boldsymbol{BC}$ and $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$

9. Given $\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, $\boldsymbol{y} = \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix}$, $\boldsymbol{z} = \begin{bmatrix} 7 & -8 & -9 \end{bmatrix}$. Find $\boldsymbol{x}^\top \boldsymbol{y}$, $\langle \boldsymbol{y}, \boldsymbol{z}^\top \rangle$ and $\boldsymbol{yz}$. Does $\langle \boldsymbol{y}, \boldsymbol{z}^\top \rangle = \boldsymbol{yz}$?

$$\boldsymbol{x}^\top \boldsymbol{y} = 2(5) + 3(6) + 4(-7) = 0$$

$$\langle \boldsymbol{y}, \boldsymbol{z}^\top \rangle = \boldsymbol{zy} = 7(5) - 8(6) - 9(-7) = 50$$

$$\boldsymbol{yz} = \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} \begin{bmatrix} 7 & -8 & -9 \end{bmatrix} = \begin{bmatrix} 35 & -40 & -45 \\ 42 & -48 & -54 \\ -49 & 56 & 63 \end{bmatrix}$$

$\langle \boldsymbol{y}, \boldsymbol{z}^\top \rangle$ is an inner product, it gives a $1-$by-1 scalar, $\boldsymbol{yz}$ gives a $3-$by-3 matrix, so $\langle \boldsymbol{y}, \boldsymbol{z}^\top \rangle \neq \boldsymbol{yz}$.

10. Compute $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 4 & -1 & 2 \\ -1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 4x - y + 2z \\ -x + z \\ 2x + y \end{bmatrix} = 4x^2 - 2xy + 4xz + 2yz$

11. Write $3x^2 - 8xy + 2y^2 + 6xz - 3z^2$ in matrix form as in the previous question.

$$3x^2 - 8xy + 2y^2 + 6xz - 3z^2 = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Diagonal terms

$$3x^2 - 8xy + 2y^2 + 6xz - 3z^2 = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 3 & ? & ? \\ ? & 2 & ? \\ ? & ? & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Off diagonal terms: coefficient / 2

$$3x^2 - 8xy + 2y^2 + 6xz - 3z^2 = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 3 & -4 & 3 \\ -4 & 2 & 0 \\ 3 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$(***)$ Actually there are infinite-many ways to do this

$$3x^2 - 8xy + 2y^2 + 6xz - 3z^2 = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 3 & -4 & 3 \\ -4 & 2 & -4 \\ 3 & 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 3 & 8 & 9999993 \\ -16 & 2 & -40 \\ -9999987 & 40 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

12. $\boldsymbol{A} = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 2 & 3 & 1 \\ 2 & -2 & -2 \\ -1 & 2 & 1 \end{bmatrix}$ $\boldsymbol{C} = \begin{bmatrix} 3 & 1 & -3 \\ 0 & 2 & 6 \\ -1 & 2 & 1 \end{bmatrix}$. Decide whether $\boldsymbol{AB} = \boldsymbol{AC}$.

$$\boldsymbol{AB} = \boldsymbol{AC} = \begin{bmatrix} 12 & 8 & 0 \\ 6 & 4 & 0 \\ -7 & -2 & 1 \end{bmatrix}$$

(What is the implication of this exercise: we have $\boldsymbol{AB} = \boldsymbol{AC}$ yet $\boldsymbol{B} \neq \boldsymbol{C}$. This shows that the cancellation law is not valid for matrix multiplication. That is, we do not have

$$\boldsymbol{AB} = \boldsymbol{AC} \quad \underbrace{\implies \boldsymbol{A}^{-1}\boldsymbol{AB} = \boldsymbol{A}^{-1}\boldsymbol{AC}}_{\text{this step is wrong}} \quad \implies \quad \boldsymbol{B} = \boldsymbol{C}$$

The reason is that $\boldsymbol{A}^{-1}$ doesn't exist.)

13. Given $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, find $\|\boldsymbol{x}\|$, $\|\boldsymbol{x}\|^2$ and the angle between $\boldsymbol{x}, \boldsymbol{y}$

$$\|\boldsymbol{x}\| = \sqrt{1^2 + 1^2} = \sqrt{2}, \qquad \|\boldsymbol{x}\|^2 = 2,$$

To find the angle between $\boldsymbol{x}, \boldsymbol{y}$ first we recall

$$\angle(\boldsymbol{x}, \boldsymbol{y}) = \cos^{-1} \frac{\boldsymbol{x}^\top \boldsymbol{y}}{\|\boldsymbol{x}\| \|\boldsymbol{y}\|} = \cos^{-1} \frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

14. Given $\boldsymbol{u} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$, find $\boldsymbol{u}^\top \boldsymbol{v}$, $\|\boldsymbol{u}\|$, $\|\boldsymbol{v}\|$ and the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$

$$\begin{aligned}
\boldsymbol{u}^\top \boldsymbol{v} &= 2(3) + 3(-1) + (-4)(-2) = 6 - 3 + 8 = 11 \\
\|\boldsymbol{u}\| &= \sqrt{2^2 + 3^2 + (-4)^2} = \sqrt{4 + 9 + 16} = \sqrt{29} \\
\|\boldsymbol{v}\| &= \sqrt{3^2 + (-1)^2 + (-2)^2} = \sqrt{9 + 1 + 4} = \sqrt{14} \\
\angle(\boldsymbol{u}, \boldsymbol{v}) &= \cos^{-1} \frac{\boldsymbol{u}^\top \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} = \cos^{-1} \frac{11}{\sqrt{29}\sqrt{14}} = \cos^{-1}(0.545920) \approx 0.577\pi = 33^\circ
\end{aligned}$$

15. Given $\boldsymbol{u} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$, is $|\boldsymbol{u}^\top \boldsymbol{v}| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$ true?

$$|\boldsymbol{u}^\top \boldsymbol{v}| = |11| = \sqrt{121} < \sqrt{406} = \sqrt{29 \cdot 14} = \|\boldsymbol{u}\|\|\boldsymbol{v}\|$$

so it is true.

16. Given $\boldsymbol{u} = \begin{bmatrix} 1 \\ -3 \\ 4 \\ 2 \end{bmatrix}$, find $\dfrac{\boldsymbol{u}}{\|\boldsymbol{u}\|}$

$$\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} = \frac{\boldsymbol{u}}{\sqrt{1 + 9 + 16 + 4}} = \frac{\boldsymbol{u}}{\sqrt{30}} = \begin{bmatrix} 1/\sqrt{30} \\ -3/\sqrt{30} \\ 4/\sqrt{30} \\ 2/\sqrt{30} \end{bmatrix}$$

17. What is the Euclidean distance between $\boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$?

$$\text{dist}(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{(1-1)^2 + (1-0)^2} = \sqrt{1} = 1$$

18. Find t such that the Euclidean distance between $\boldsymbol{u} = \begin{bmatrix} 1 \\ t \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is 2.

$$\text{dist}(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{(1-2)^2 + (3-t)^2} = 2 \iff \sqrt{1 + 9 - 6t + t^2} = 2 \iff t^2 - 6t + 6 = 0$$

$$t = \frac{6 \pm \sqrt{36 - 4(6)}}{2} = \frac{6 \pm \sqrt{36 - 4(6)}}{2} = 3 \pm \sqrt{3}$$

19. Find t such that if $\boldsymbol{A} = \begin{bmatrix} 1 & t \\ 3 & 1 \end{bmatrix}$, we have $\boldsymbol{A}^\top = \boldsymbol{A}$

$t = 3$

20. Given $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 0 \\ -3 & -3 & -3 \\ 0 & 0 & 4 \end{bmatrix}$, find $\boldsymbol{A}^\top$, $\boldsymbol{A}\boldsymbol{A}^\top$ and $\boldsymbol{A}^\top\boldsymbol{A}$, is $\boldsymbol{A}\boldsymbol{A}^\top = \boldsymbol{A}^\top\boldsymbol{A}$?

$$\boldsymbol{A}^\top = \begin{bmatrix} 1 & -3 & 0 \\ 2 & -3 & 0 \\ 0 & -3 & 4 \end{bmatrix} \quad \boldsymbol{A}\boldsymbol{A}^\top = \begin{bmatrix} 5 & -9 & 0 \\ -9 & 27 & -12 \\ 0 & -12 & 16 \end{bmatrix} \quad \boldsymbol{A}^\top\boldsymbol{A} = \begin{bmatrix} 10 & 11 & 9 \\ 11 & 13 & 9 \\ 9 & 9 & 25 \end{bmatrix}$$

21. Given $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, find $(\boldsymbol{A}\boldsymbol{B})^\top$, $\boldsymbol{B}^\top\boldsymbol{A}^\top$, is $(\boldsymbol{A}\boldsymbol{B})^\top = \boldsymbol{B}^\top\boldsymbol{A}^\top$?

$$\boldsymbol{A}\boldsymbol{B} = \begin{bmatrix} 5 & 2 \\ 6 & 3 \end{bmatrix}, \quad (\boldsymbol{A}\boldsymbol{B})^\top = \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix} = \boldsymbol{B}^\top\boldsymbol{A}^\top = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

22. A matrix $\boldsymbol{A}$ is called orthogonal if $\boldsymbol{A}\boldsymbol{A}^\top = \boldsymbol{I}$. Which of the following matrices is orthogonal?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}$$

All matrices are orthogonal except the last one.

23. Given $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. Find $\boldsymbol{A}^2, \boldsymbol{A}^3, \boldsymbol{A}^4$, do you find something?

$$\boldsymbol{A}^n = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$$

How to get this

$$\begin{aligned} \boldsymbol{A}^2 &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\ &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

and this is true for $\boldsymbol{A}^3, \boldsymbol{A}^4$, so basically the effect of multiplying $\boldsymbol{A}$ is to add 2 on the top-right entry

24. True or false: the diagonal of a matrix $\boldsymbol{A}$ and the diagonal of a matrix $\boldsymbol{A}^\top$ are always the same for any square matrix $\boldsymbol{A}$ true

25. $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 3 \end{bmatrix}$, turn $\boldsymbol{A}$ into a row echelon form.

Replace row2 by -2row1 + row2 and replace row3 by -3row1+row3 gives

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 5 & 3 \end{bmatrix}$$

Replace row3 by -5row2+4row3

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

26. $\boldsymbol{A} = \begin{bmatrix} -4 & 1 & -6 \\ 1 & 2 & -5 \\ 6 & 3 & -4 \end{bmatrix}$, turn $\boldsymbol{A}$ into a row echelon form.

swap row1 and row2

$$\begin{bmatrix} 1 & 2 & -5 \\ -4 & 1 & -6 \\ 6 & 3 & -4 \end{bmatrix}$$

replace row2 by 4row1 + row2 and replace row3 by -6row1 + row3

$$\begin{bmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & -9 & 26 \end{bmatrix}$$

replace row3 by row2 + row3

$$\begin{bmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & 0 & 0 \end{bmatrix}$$

27. $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -4 & -4 \\ 2 & 5 & -9 & -10 \\ 3 & -2 & 3 & 11 \end{bmatrix}$, turn $\boldsymbol{A}$ into a row echelon form.

replace row2 by -2row1+row2 and replace row3 by -3row1 + row3

$$\begin{bmatrix} 1 & 2 & -4 & -4 \\ 0 & 1 & -1 & -2 \\ 0 & -8 & 15 & 23 \end{bmatrix}$$

replace row3 by 8row2 + row3

$$\begin{bmatrix} 1 & 2 & -4 & -4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 7 & 7 \end{bmatrix}$$

4.3 Determinant

1. Find the determinant of the following matrices.

$$\boldsymbol{A} = \begin{bmatrix}0\end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix}1 & -2 & 3\\2 & 0 & 3\\1 & -2 & 3\end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix}1 & 0 & 0 & 0\\2 & 1 & 2 & 3\\1 & 4 & 5 & 6\\3 & 7 & 8 & 9\end{bmatrix}$$

2. Find $\det\begin{bmatrix}-7 & -10 & 4\\3 & -9 & 2\\7 & 1 & 2\end{bmatrix}$ by Laplace's cofactor expansion with clever use of the property of determinant

3. Find $\det\begin{bmatrix}1 & 0 & -3 & 0 & 9\\3 & 7 & 10 & 3 & 17\\4 & 0 & 11 & 0 & 1\\6 & 0 & 8 & 0 & -3\\5 & 1 & 6 & -1 & 8\end{bmatrix}$. Hint: look at the 4th column.

4. $\boldsymbol{A} = \begin{bmatrix}1 & 1\\2 & 2\end{bmatrix}$, is $\boldsymbol{A}$ non-singular? Why?

5. Find $\det\begin{bmatrix}1 & 2 & 3\\4 & -2 & 3\\0 & 5 & -1\end{bmatrix}$

6. Find $\det\begin{bmatrix}1 & 1 & -3\\1 & 1 & 10\\1 & 1 & 11\end{bmatrix}$.

7. Find $\det\begin{bmatrix}1 & 1 & -3\\0 & 2 & 10\\0 & 0 & 3\end{bmatrix}$.

8. Find $\det\begin{bmatrix}2 & 6\\4 & 1\end{bmatrix}$.

9. Find t such that $\det\begin{bmatrix}t-4 & 3\\2 & t-9\end{bmatrix} = 0$.

10. True or false: $\det\boldsymbol{A} = \det\boldsymbol{A}^\top$

11. True or false: if $\boldsymbol{A}$ has a zero column, then $\det\boldsymbol{A} = 0$

12. True or false: $\det(\boldsymbol{AB}) = (\det\boldsymbol{A})(\det\boldsymbol{B})$

13. True or false: $\det(\boldsymbol{A}-\boldsymbol{B}) = \det\boldsymbol{A} - \det\boldsymbol{B}$

14. Find the characteristic polynomial of $\boldsymbol{A} = \begin{bmatrix}5 & 3\\2 & 10\end{bmatrix}$

15. Find the characteristic polynomial of $\boldsymbol{A} = \begin{bmatrix}7 & -1\\6 & 2\end{bmatrix}$

16. Find the characteristic polynomial of $\boldsymbol{A} = \begin{bmatrix}5 & -2\\4 & -4\end{bmatrix}$

17. Find the characteristic polynomial of $\boldsymbol{A} = \begin{bmatrix}3 & -2\\9 & -3\end{bmatrix}$

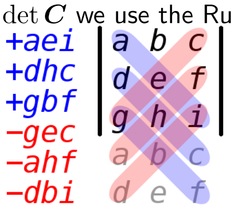
18. Find the characteristic polynomial of $\boldsymbol{A} = \begin{bmatrix}1 & 2 & 3\\3 & 0 & 4\\6 & 4 & 5\end{bmatrix}$

4.4 Determinant solution

1. Find the determinant of the following matrices.

$$\boldsymbol{A} = \begin{bmatrix}0\end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix}1 & -2 & 3\\2 & 0 & 3\\1 & -2 & 3\end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix}1 & 0 & 0 & 0\\2 & 1 & 2 & 3\\1 & 4 & 5 & 6\\3 & 7 & 8 & 9\end{bmatrix}$$

- $\det \boldsymbol{A} = 0$, determinant of scalar = the scalar itself
- $\det \boldsymbol{B} = 4 - 6 = -2$ because $\det \begin{bmatrix}a & b\\c & d\end{bmatrix} = ad - bc$
- $\det \boldsymbol{C}$ we use the Rule of Sarrus



$$\begin{aligned}\det \boldsymbol{C} &= 1(5)(9) + 2(6)(7) + 3(4)(8) - 1(8)(6) - 4(2)(9) - 7(5)(3)\\ &= 45 + 84 + 96 - 48 - 72 - 105\\ &= 0\end{aligned}$$

- $\det \boldsymbol{D} = 0$ because 1st-row = 3rd row
- For $\det \boldsymbol{E}$ we use cofactor $\det \boldsymbol{E} = 1 \times \det \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix} = 0$

2. Find $\det \begin{bmatrix}-7 & -10 & 4\\3 & -9 & 2\\7 & 1 & 2\end{bmatrix}$ by Laplace's cofactor expansion with clever use of the property of determinant

$$\det \begin{bmatrix}-7 & -10 & 4\\3 & -9 & 2\\7 & 1 & 2\end{bmatrix} = \det \begin{bmatrix}-7 & -10 & 4\\3 & -9 & 2\\0 & -9 & 6\end{bmatrix} = 9 \det \begin{bmatrix}-7 & 4\\3 & 2\end{bmatrix} + 6 \det \begin{bmatrix}-7 & -10\\3 & -9\end{bmatrix} = 9(-14-12) + 6(63+30) = 324$$

(you can use any way to solve it as long as you get 324)

3. Find $\det \begin{bmatrix}1 & 0 & -3 & 0 & 9\\3 & 7 & 10 & 3 & 17\\4 & 0 & 11 & 0 & 1\\6 & 0 & 8 & 0 & -3\\5 & 1 & 6 & -1 & 8\end{bmatrix}$. Hint: look at the 4th column.

Using the property of determinant: added 3 times the last row to the second row do not change the determinant, so

$$\det \begin{bmatrix}1 & 0 & -3 & 0 & 9\\3 & 7 & 10 & 3 & 17\\4 & 0 & 11 & 0 & 1\\6 & 0 & 8 & 0 & -3\\5 & 1 & 6 & -1 & 8\end{bmatrix} = \det \begin{bmatrix}1 & 0 & -3 & 0 & 9\\18 & 10 & 28 & 0 & 41\\4 & 0 & 11 & 0 & 1\\6 & 0 & 8 & 0 & -3\\5 & 1 & 6 & -1 & 8\end{bmatrix}$$

Now, we develop the determinant about the fourth column:

$$\det \begin{bmatrix}1 & 0 & -3 & 0 & 9\\18 & 10 & 28 & 0 & 41\\4 & 0 & 11 & 0 & 1\\6 & 0 & 8 & 0 & -3\\5 & 1 & 6 & -1 & 8\end{bmatrix} = (-1)(-1)^{4+5} \det \begin{bmatrix}1 & 0 & -3 & 9\\18 & 10 & 28 & 41\\4 & 0 & 11 & 1\\6 & 0 & 8 & -3\end{bmatrix}$$

Now note that 2nd column has only one nonzero, so we develop the determinant about the 2nd column

$$\det \begin{bmatrix}1 & 0 & -3 & 9\\18 & 10 & 28 & 41\\4 & 0 & 11 & 1\\6 & 0 & 8 & -3\end{bmatrix} = (10)(-1)^{2+2} \det \begin{bmatrix}1 & -3 & 9\\4 & 11 & 1\\6 & 8 & -3\end{bmatrix} = 10(-33 - 18 + 288 - 594 - 8 - 36) = -4010$$

4. $\boldsymbol{A} = \begin{bmatrix}1 & 1\\2 & 2\end{bmatrix}$, is $\boldsymbol{A}$ non-singular? Why?

$\det \boldsymbol{A} = 0$ so $\boldsymbol{A}$ is singular $\iff$ $\boldsymbol{A}^{-1}$ does not exist.

5. Find $\det \begin{bmatrix}1 & 2 & 3\\4 & -2 & 3\\0 & 5 & -1\end{bmatrix}$

There is a zero so we use Laplace expansion along the first column

$$\det \begin{bmatrix}1 & 2 & 3\\4 & -2 & 3\\0 & 5 & -1\end{bmatrix} = \det \begin{bmatrix}-2 & 3\\5 & -1\end{bmatrix} - 4 \det \begin{bmatrix}2 & 3\\5 & -1\end{bmatrix} = (2-15) - 4(-2-15) = 55$$

6. Find $\det \begin{bmatrix} 1 & 1 & -3 \\ 1 & 1 & 10 \\ 1 & 1 & 11 \end{bmatrix}$.
   It is zero.

7. Find $\det \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 10 \\ 0 & 0 & 3 \end{bmatrix}$.
   The product of diagonal = 1(2)(3) = 6

8. Find $\det \begin{bmatrix} 2 & 6 \\ 4 & 1 \end{bmatrix}$.
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9. Find t such that $\det \begin{bmatrix} t-4 & 3 \\ 2 & t-9 \end{bmatrix} = 0$.
   $t = 3$ or 10

10. True or false: $\det \boldsymbol{A} = \det \boldsymbol{A}^\top$ true

11. True or false: if $\boldsymbol{A}$ has a zero column, then $\det \boldsymbol{A} = 0$ true

12. True or false: $\det(\boldsymbol{AB}) = (\det \boldsymbol{A})(\det \boldsymbol{B})$ true

13. True or false: $\det(\boldsymbol{A} - \boldsymbol{B}) = \det \boldsymbol{A} - \det \boldsymbol{B}$ false

14. Find the characteristic polynomial of $\boldsymbol{A} = \begin{bmatrix} 5 & 3 \\ 2 & 10 \end{bmatrix}$

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \lambda^2 - 15\lambda + 44$$

15. Find the characteristic polynomial of $\boldsymbol{A} = \begin{bmatrix} 7 & -1 \\ 6 & 2 \end{bmatrix}$

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \lambda^2 - 9\lambda + 20$$

16. Find the characteristic polynomial of $\boldsymbol{A} = \begin{bmatrix} 5 & -2 \\ 4 & -4 \end{bmatrix}$

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \lambda^2 - \lambda - 12$$

17. Find the characteristic polynomial of $\boldsymbol{A} = \begin{bmatrix} 3 & -2 \\ 9 & -3 \end{bmatrix}$

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \lambda^2 + 9$$

18. Find the characteristic polynomial of $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 4 \\ 6 & 4 & 5 \end{bmatrix}$

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \lambda^3 - 6\lambda^2 - 35\lambda - 38$$

4.5 Inverse

1. $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, find the formula of $\boldsymbol{A}^{-1}$
2. Let $\boldsymbol{A} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & 2 \\ -3 & -4 & -4 \end{bmatrix}$, find $\boldsymbol{A}^{-1}$
3. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, find $\boldsymbol{A}^{-1}$
4. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, find $\boldsymbol{A}^{-1}$
5. Given $\boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $\boldsymbol{B}\boldsymbol{A}^2 = \boldsymbol{A}$, find $\boldsymbol{B}$
6. Find the inverse of $\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

4.6 Inverse solution

1. $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, find the formula of $\boldsymbol{A}^{-1}$

$$\boldsymbol{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

2. Let $\boldsymbol{A} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & 2 \\ -3 & -4 & -4 \end{bmatrix}$, find $\boldsymbol{A}^{-1}$.

$$\boldsymbol{A}^{-1} = \frac{1}{30}\begin{bmatrix} 0 & -20 & -10 \\ -6 & 5 & -2 \\ 6 & 10 & 2 \end{bmatrix}$$

3. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, find $\boldsymbol{A}^{-1}$

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

4. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, find $\boldsymbol{A}^{-1}$

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & -2/3 \\ 0 & 1/3 \end{bmatrix}$$

5. Given $\boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $\boldsymbol{B}\boldsymbol{A}^2 = \boldsymbol{A}$, find $\boldsymbol{B}$

$$\det \boldsymbol{A} = \det \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} = 1 \neq 1 \quad \Longrightarrow \boldsymbol{A}^{-1} \text{ exists and } \boldsymbol{A}^{-1} = \frac{1}{1}\begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

$$\boldsymbol{B}\boldsymbol{A}^2 = \boldsymbol{A} \iff \boldsymbol{B}\boldsymbol{A}^2\boldsymbol{A}^{-1} = \boldsymbol{A}\boldsymbol{A}^{-1} \iff \boldsymbol{B}\boldsymbol{A} = \boldsymbol{I} \iff \boldsymbol{B}\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1} \iff \boldsymbol{B}\boldsymbol{I} = \boldsymbol{A}^{-1} \iff \boldsymbol{B} = \boldsymbol{A}^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

6. Find the inverse of $\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

$$\boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}}\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} = \frac{-1}{2}\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}$$

4.7 Eigenvector and eigenvalue

1. Find the eigenvalues of $\boldsymbol{A} = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$

2. Find the eigenvectors of $\boldsymbol{A} = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$

3. Find the eigenvalues and eigenvectors of $\boldsymbol{A} = \begin{bmatrix} 3 & -4 \\ 2 & -6 \end{bmatrix}$

4. If I give you a matrix $\boldsymbol{A}$ and a vector $\boldsymbol{x}$, how to tell is $\boldsymbol{x}$ an eigenvector of $\boldsymbol{A}$?

5. Find a matrix $\boldsymbol{A}$ whose eigenvalues are 1 and 4, and whose eigenvectors are $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, respectively.

6. Find a matrix $\boldsymbol{A}$ whose eigenvalues are 1 and 1, and whose eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively.

7. Let $\boldsymbol{A} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$, find $\boldsymbol{A}^6$ using eigendecomposition.

8. Let $\boldsymbol{A} = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$, find $\boldsymbol{A}^8$ using eigendecomposition.

9. $\boldsymbol{A} = \begin{bmatrix} 4 & 0 & -2 \\ 1 & 3 & -2 \\ 1 & 2 & -1 \end{bmatrix}$, find the characteristic polynomial of $\boldsymbol{A}$ and find all eigenvalues.

10. $\boldsymbol{A} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$, find the characteristic polynomial of $\boldsymbol{A}$. Given 3 is an eigenvalue, find all the other eigenvalues.

4.8 Eigenvector and eigenvalue solution

1. Find the eigenvalues of $\boldsymbol{A} = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$

$$\det(\lambda \boldsymbol{I}_2 - \boldsymbol{A}) = 0 \iff \det \begin{bmatrix} \lambda+5 & -2 \\ 7 & \lambda-4 \end{bmatrix} = 0 \iff \lambda^2 + \lambda - 6 = 0 \iff \lambda_1 = 2, \lambda_2 = -3$$

2. Find the eigenvectors of $\boldsymbol{A} = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x} \iff (\lambda_1 \boldsymbol{I}_2 - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0} \iff \begin{bmatrix} \lambda+5 & -2 \\ 7 & \lambda-4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 = 2 \implies \begin{bmatrix} 7 & -2 \\ 7 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & -2/7 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \boldsymbol{v}_1 = t\begin{bmatrix} 2/7 \\ 1 \end{bmatrix}, t \in \mathbb{R} \implies \boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

$$\lambda_1 = -3 \implies \begin{bmatrix} 2 & -2 \\ 7 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \boldsymbol{v}_2 = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}, t \in \mathbb{R} \implies \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

3. Find the eigenvalues and eigenvectors of $\boldsymbol{A} = \begin{bmatrix} 3 & -4 \\ 2 & -6 \end{bmatrix}$

$$\lambda_1 = 2, \quad \lambda_2 = 5, \quad \boldsymbol{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

4. If I give you a matrix $\boldsymbol{A}$ and a vector $\boldsymbol{x}$, how to tell is $\boldsymbol{x}$ an eigenvector of $\boldsymbol{A}$?
Check is $\boldsymbol{A}\boldsymbol{x} = k\boldsymbol{x}$ for some number k

5. Find a matrix $\boldsymbol{A}$ whose eigenvalues are 1 and 4, and whose eigenvectors are $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, respectively.

We have $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$, then

$$\boldsymbol{A}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \boldsymbol{A}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

Since everything is in 2-dimensional, then $\boldsymbol{A}$ is a 2-by-2 matrix and thus

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

The two equations have to be hold at the same time, meaning we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 1 & 4 \end{bmatrix}$$

Thus

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 8 \\ 1 & 4 \end{bmatrix}\frac{1}{1}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 18 \\ -3 & 10 \end{bmatrix}$$

6. Find a matrix $\boldsymbol{A}$ whose eigenvalues are 1 and 1, and whose eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively.

$$\boldsymbol{A} = \boldsymbol{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

7. Let $\boldsymbol{A} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$, find $\boldsymbol{A}^6$ using eigendecomposition.

- The characteristic polynomial is $\lambda^2 - 5\lambda + 4$
- The eigenvalues $\lambda_1 = 1, \lambda_2 = 4$
- The eigenvectors are $\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 2/3 \end{bmatrix}$
- $\boldsymbol{A}^6 = \boldsymbol{V}\boldsymbol{D}^6\boldsymbol{V}^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4^6 \end{bmatrix}\begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4096 \end{bmatrix}\begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 1336 & 2230 \\ 1365 & 2731 \end{bmatrix}$

8. Let $\boldsymbol{A} = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$, find $\boldsymbol{A}^8$ using eigendecomposition.

- The eigenvalues $\lambda_1 = 1, \lambda_2 = 4$
- The eigenvectors are $\boldsymbol{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix}1\\-2\end{bmatrix}$
- $\boldsymbol{A}^8 = \boldsymbol{V}\boldsymbol{D}^8\boldsymbol{V}^{-1} = \begin{bmatrix}21846 & -21845\\-43690 & 43691\end{bmatrix}$

9. $\boldsymbol{A} = \begin{bmatrix}4 & 0 & -2\\1 & 3 & -2\\1 & 2 & -1\end{bmatrix}$, find the characteristic polynomial of $\boldsymbol{A}$ and find all eigenvalues.

$$p(\lambda) = \det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \det\begin{bmatrix}4-\lambda & 0 & -2\\1 & 3-\lambda & -2\\1 & 2 & -1-\lambda\end{bmatrix}$$

We perform Laplace expansion along the first row

$$\begin{aligned}\det\begin{bmatrix}4-\lambda & 0 & -2\\1 & 3-\lambda & -2\\1 & 2 & -1-\lambda\end{bmatrix} &= (4-\lambda)(-1)^{1+1}\det\begin{bmatrix}3-\lambda & -2\\2 & -1-\lambda\end{bmatrix} + 0 + (-2)(-1)^{1+3}\det\begin{bmatrix}1 & 3-\lambda\\1 & 2\end{bmatrix}\\ &= (4-\lambda)\det\begin{bmatrix}3-\lambda & -2\\2 & -1-\lambda\end{bmatrix} - 2\det\begin{bmatrix}1 & 3-\lambda\\1 & 2\end{bmatrix}\\ &= (4-\lambda)\Big((3-\lambda)(-1-\lambda)+4\Big) - 2\Big(2-(3-\lambda)\Big)\\ &= -\lambda^3 + 6\lambda^2 - 11\lambda + 6.\end{aligned}$$

So the characteristic polynomial of $p(\lambda) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6$

Now to find the eigenvalues, we solve $p(\lambda) = 0$

$$\begin{aligned}& -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0\\ \iff\ & \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0\\ \iff\ & (\lambda-1)(\lambda-2)(\lambda-3) = 0 \quad \text{(this factorization may be not easy for some students)}\end{aligned}$$

Therefore, the eigenvalues are $1, 2, 3$.

10. $\boldsymbol{A} = \begin{bmatrix}4 & 1 & -1\\2 & 5 & -2\\1 & 1 & 2\end{bmatrix}$, find the characteristic polynomial of $\boldsymbol{A}$. Given 3 is an eigenvalue, find all the other eigenvalues.

$$\det(\lambda\boldsymbol{I} - \boldsymbol{A}) = \lambda^3 - 11\lambda^2 + 39\lambda - 45$$

3 is an eigenvalue, so $(\lambda - 3)$ is a factor in the polynomial. Dividing $\lambda^3 - 11\lambda^2 + 39\lambda - 45$ by $\lambda - 3$ gives $\lambda^2 - 8\lambda + 15$

$$\lambda^3 - 11\lambda^2 + 39\lambda - 45 = (\lambda-3)(\lambda^2 - 8\lambda + 15) = (\lambda-3)(\lambda-3)(\lambda-5) = (\lambda-3)^2(\lambda-5)$$

So the eigenvalues of $\boldsymbol{A}$ are 3 (2 times) and 5

(If you don't know how to divide polynomial, see `https://en.wikipedia.org/wiki/Synthetic_division`)

4.9 System of equations

1. Solve $\begin{cases} x + 2y - z = 6 \\ 3x + 8y + 9z = 10 \\ 2x - y + 2z = -2 \end{cases}$

2. Solve $\begin{cases} x + y + 2z = 6 \\ 3x + 4y - z = 5 \\ -x + y + z = 2 \end{cases}$

3. Solve $\begin{cases} 2x + y = 5 \\ 3x + 6y + z = 1 \\ 2x + 7y + z = 8 \end{cases}$

4. Solve $\begin{cases} -3x + 6y - z + w = -7 \\ x - 2y + 2z + 3w = -1 \\ 2x - 4y + 5z + 8w = -4 \end{cases}$

5. Solve $\begin{cases} 2x + 2y - z = 1 \\ -2x - 2y + 4z = 1 \\ 2x + 2y + 5z = 5 \\ -2x - 2y - 2z = 3 \end{cases}$

6. Solve $\begin{cases} x + 2y - z = 0 \\ 3x + 8y + 2z = 0 \\ 4x + 9y - z = 0 \end{cases}$

7. What is the relationship between system of linear equations and determinant?

8. For $\begin{cases} 2x + 5y + 3z = 2 \\ x + 2y + 2z = 4 \\ x + y + 3z = 10 \end{cases}$ Is this system has a unique solution? If yes, why? If no, why?

9. For $\begin{cases} x + y - 2z = 2 \\ 3x - y + 6z = 2 \\ 6x + 5y - 9z = 3 \end{cases}$ Is this system has a unique solution? If yes, why? If no, why?

10. Find all solutions of the inhomogeneous system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A} := \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 2 \end{bmatrix}, \quad \boldsymbol{b} := \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

11. Find all solutions of the inhomogeneous system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A} := \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \end{bmatrix}, \quad \boldsymbol{b} := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

4.10 System of equations solution

1. Solve $\begin{cases} x+2y-z=6 \\ 3x+8y+9z=10 \\ 2x-y+2z=-2 \end{cases}$

$$\begin{cases} x+2y-z=6 \\ 3x+8y+9z=10 \\ 2x-y+2z=-2 \end{cases} \iff \begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 9 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ -2 \end{bmatrix}$$

2nd-row adds -3 times of 1st-row

$$\iff \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 9 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 10 \\ -2 \end{bmatrix} \iff \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 12 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -2 \end{bmatrix}$$

3rd-row adds -2 times of 1st-row

$$\iff \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 12 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ -8 \\ -2 \end{bmatrix} \iff \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 12 \\ 0 & -5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -14 \end{bmatrix}$$

divide 2nd-row by 2 (multiply 2nd-row by 0.5)

$$\iff \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 12 \\ 0 & -5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ -8 \\ -14 \end{bmatrix} \iff \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 6 \\ 0 & -5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ -14 \end{bmatrix}$$

3rd-row adds 5 times 2nd-row

$$\iff \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 6 \\ 0 & -5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ -4 \\ -14 \end{bmatrix} \iff \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 6 \\ 0 & 0 & 34 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ -34 \end{bmatrix}$$

divide 3rd-row by 34 (multiply 3rd-row by 1/34)

$$\iff \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/34 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 6 \\ 0 & 0 & 34 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/34 \end{bmatrix} \begin{bmatrix} 6 \\ -4 \\ -34 \end{bmatrix} \iff \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ -1 \end{bmatrix}$$

Backward substitution: $z=-1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix} \iff y+6z=-4 \iff y-6=-4 \iff y=2$$

$$\begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \end{bmatrix} \iff x+2y-z=6 \iff x+4+1=6 \iff x=1$$

2. Solve $\begin{cases} x+y+2z=6 \\ 3x+4y-z=5 \\ -x+y+z=2 \end{cases}$

$$\begin{cases} x+y+2z=6 \\ 3x+4y-z=5 \\ -x+y+z=2 \end{cases} \iff \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 2 \end{bmatrix}$$

-3(1st-row) add to 2nd-row, 1st-row add to 3rd-row

$$\begin{bmatrix} 1 & & \\ -3 & 1 & \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & & \\ -3 & 1 & \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 2 \end{bmatrix} \iff \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -7 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -13 \\ 8 \end{bmatrix}$$

-2(2nd-row) add to 3rd-row

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -7 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -2 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ -13 \\ 8 \end{bmatrix} \iff \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -7 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -13 \\ 34 \end{bmatrix}$$

$17z=34 \implies z=2$

$$\begin{cases} x+y+2z=6 \\ y-7z=-13 \\ z=2 \end{cases} \implies \begin{cases} x+y+4=6 \\ y-14=-13 \\ z=2 \end{cases} \implies \begin{cases} x+y=2 \\ y=1 \\ z=2 \end{cases} \implies \begin{cases} x=1 \\ y=1 \\ z=2 \end{cases}$$

3. Solve $\begin{cases} 2x+y=5 \\ 3x+6y+z=1 \\ 2x+7y+z=8 \end{cases}$

$$\begin{cases} 2x+y=5 \\ 3x+6y+z=1 \\ 2x+7y+z=8 \end{cases} \iff \begin{bmatrix} 2 & 1 & 0 \\ 3 & 6 & 1 \\ 5 & 7 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 8 \end{bmatrix}$$

half the 1st-row, one-third the 2nd-row, one-fifth the 3rd-row

$$\iff \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 6 & 1 \\ 5 & 7 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 8 \end{bmatrix} \iff \begin{bmatrix} 1 & 1/2 & 0 \\ 1 & 2 & 1/3 \\ 1 & 7/5 & 1/5 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1/3 \\ 8/5 \end{bmatrix}$$

2nd-row add -1 times 1st-row, 3rd-row add -1 times 1st-row

$$\iff \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 0 \\ 1 & 2 & 1/3 \\ 1 & 7/5 & 1/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5/2 \\ 1/3 \\ 8/5 \end{bmatrix} \iff \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 2-1/2 & 1/3 \\ 0 & 7/5-1/2 & 1/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1/3-5/2 \\ 8/5-5/2 \end{bmatrix}$$

$$\iff \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 3/2 & 1/3 \\ 0 & 9/10 & 1/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5/2 \\ -13/6 \\ -9/10 \end{bmatrix} \iff$$

1st-row times 2, 2nd-row times 6, 3rd-row times 10

$$\iff \begin{bmatrix} 2 & & \\ & 6 & \\ & & 10 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 3/2 & 1/3 \\ 0 & 9/10 & 1/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & & \\ & 6 & \\ & & 10 \end{bmatrix} \begin{bmatrix} 5/2 \\ -13/6 \\ -9/10 \end{bmatrix} \iff \begin{bmatrix} 2 & 1 & 0 \\ 0 & 9 & 2 \\ 0 & 9 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -13 \\ -9 \end{bmatrix}$$

3rd-row add -1 times 2nd-row

$$\iff \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 9 & 2 \\ 0 & 9 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -13 \\ -9 \end{bmatrix} \iff \begin{bmatrix} 2 & 1 & 0 \\ 0 & 9 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -13 \\ 4 \end{bmatrix}$$

No solution.

4. Solve $\begin{cases} -3x+6y-z+w=-7 \\ x-2y+2z+3w=-1 \\ 2x-4y+5z+8w=-4 \end{cases}$

$$\begin{cases} -3x+6y-z+w=-7 \\ x-2y+2z+3w=-1 \\ 2x-4y+5z+8w=-4 \end{cases} \iff \begin{bmatrix} -3 & 6 & -1 & 1 \\ 1 & -2 & 2 & 3 \\ 2 & -4 & 5 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -7 \\ -1 \\ -4 \end{bmatrix}$$

-2(2nd-row) add to 3rd-row, 3(2nd-row) add to 1st-row

$$\begin{bmatrix} 1 & 3 & \\ & 1 & \\ & -2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 6 & -1 & 1 \\ 1 & -2 & 2 & 3 \\ 2 & -4 & 5 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 & 3 & \\ & 1 & \\ & -2 & 1 \end{bmatrix} \begin{bmatrix} -7 \\ -1 \\ -4 \end{bmatrix} \iff \begin{bmatrix} 0 & 0 & 5 & 10 \\ 1 & -2 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -7-3 \\ -1 \\ 2-4 \end{bmatrix}$$

$-\frac{2}{5}$(1st-row) add to 2nd-row, $-\frac{1}{5}$(1st-row) add to 3rd-row,

$$\begin{bmatrix} 1 & & \\ -2/5 & 1 & \\ -1/5 & & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 5 & 10 \\ 1 & -2 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 & & \\ -2/5 & 1 & \\ -1/5 & & 1 \end{bmatrix} \begin{bmatrix} -10 \\ -1 \\ -2 \end{bmatrix} \iff \begin{bmatrix} 0 & 0 & 5 & 10 \\ 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -10 \\ 3 \\ 0 \end{bmatrix}$$

Swap (1/5)(1st-row) and 2nd-row

$$\begin{bmatrix} & 1 & \\ 1/5 & & \\ & & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 5 & 10 \\ 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} & 1 & \\ 1/5 & & \\ & & 1 \end{bmatrix} \begin{bmatrix} -10 \\ 3 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$$

We have $\begin{cases} x - 2y - w = 3 \\ z + 2w = -2 \end{cases}$, more variables than number of equations $\implies$ infinitely-many solution.
To get the solution set, we write z as a function of w and write x as a function of y and w:

$$\left\{ \begin{bmatrix} 3+2y+w \\ y \\ -2-2w \\ w \end{bmatrix} \ : \ y \in \mathbb{R}, \ w \in \mathbb{R}, \right\}$$

5. Solve $\begin{cases} 2x + 2y - z = 1 \\ -2x - 2y + 4z = 1 \\ 2x + 2y + 5z = 5 \\ -2x - 2y - 2z = 3 \end{cases}$ Solution: $x = \dfrac{5}{6} - y, \quad z = \dfrac{2}{3}, \ y \in \mathbb{R}$

6. Solve $\begin{cases} x + 2y - z = 0 \\ 3x + 8y + 2z = 0 \\ 4x + 9y - z = 0 \end{cases}$ Solution: $x = y = z = 0$

7. What is the relationship between system of linear equations and determinant?

   - $\det \boldsymbol{A} = 0 \iff \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has $\begin{cases} \text{no solution, or} \\ \text{solution but infinitely many} \end{cases} \iff$ no unique solution.
   - $\det \boldsymbol{A} \neq 0 \iff \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has an unique solution.

8. For $\begin{cases} 2x + 5y + 3z = 2 \\ x + 2y + 2z = 4 \\ x + y + 3z = 10 \end{cases}$ Is this system has a unique solution? If yes, why? If no, why?

$$\det \begin{bmatrix} 2 & 5 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 3 \end{bmatrix} = 2(2)(3) + 5(2)(1) + 3 - 6 - 2(2) - 5(3) = 12 + 10 + 3 - 6 - 4 - 15 = 0$$

$\det \boldsymbol{A} = 0 \iff \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has $\begin{cases} \text{no solution, or} \\ \text{solution but infinitely many} \end{cases} \iff$ no unique solution.

9. For $\begin{cases} x + y - 2z = 2 \\ 3x - y + 6z = 2 \\ 6x + 5y - 9z = 3 \end{cases}$ Is this system has a unique solution? If yes, why? If no, why?

this time we calculate $\det \begin{bmatrix} 1 & 1 & -2 \\ 3 & -1 & 6 \\ 6 & 5 & -9 \end{bmatrix}$ using cofactor of the first row

$$\det \begin{bmatrix} 1 & 1 & -2 \\ 3 & -1 & 6 \\ 6 & 5 & -9 \end{bmatrix} = 1 \cdot \det \begin{bmatrix} -1 & 6 \\ 5 & -9 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} 3 & 6 \\ 6 & -9 \end{bmatrix} + (-2) \cdot \det \begin{bmatrix} 3 & -1 \\ 6 & 5 \end{bmatrix} = 9 - 30 - (-27 - 36) - 2(15 + 6) \ = \ 0$$

$\det \boldsymbol{A} = 0 \iff \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has $\begin{cases} \text{no solution, or} \\ \text{solution but infinitely many} \end{cases} \iff$ no unique solution.

10. Find all solutions of the inhomogeneous system of linear equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{A} := \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 2 \end{bmatrix}, \quad \boldsymbol{b} := \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 0 & 0 \\ -1 & 2 & 1 \end{array}\right] \longrightarrow \begin{array}{l} \text{Row2 - 3Row1} \\ \text{Row3 + Row1} \end{array} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -6 & -3 \\ 0 & 4 & 2 \end{array}\right] \longrightarrow \begin{array}{l} \text{Row1 + 1/3 Row2} \\ \text{Row2 /6} \\ \text{Row3 + 2/3 Row2} \end{array} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right]$$

From the last row of this augmented system, we see that $0x_1 + 0x_2 = 0$, which is always true. From the other rows, we obtain $x_1 = 0$ and $x_2 = \dfrac{1}{2}$, so that

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

is the unique solution of the system of linear equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

11. Find all solutions of the inhomogeneous system of linear equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{A} := \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \end{bmatrix}, \quad \boldsymbol{b} := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The general solution consists of a particular solution of the inhomogeneous system and all solutions of the homogeneous system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$. An efficient way to determine the general solution is via the reduced row echelon form (RREF) of the augmented system $[\boldsymbol{A}|\boldsymbol{b}]$:

$$\begin{bmatrix} 1 & 2 & 3 & | & 1 \\ 0 & 2 & 2 & | & 1 \end{bmatrix} \quad \longrightarrow \quad \begin{matrix} \text{Row1 - Row2} \\ \text{Row2 /2} \end{matrix} \begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & 1 & | & 1/2 \end{bmatrix}$$

- From the RREF, we can read out a *particular solution* (not unique) by using the pivot columns as

$$\boldsymbol{x}_p = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} \in \mathbb{R}^3 .$$

  Here, we set x_1 to the right-hand side of the augmented RREF in the first row, and x_2 to the right-hand side of the augmented RREF in the second row. Since $\boldsymbol{x}_p \in \mathbb{R}^3$ (otherwise the matrix-vector multiplication $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ would not be defined), the third coordinate $x_3 = 0$.

- Next, we determine all solutions of the homogeneous system of linear equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$. From the left-hand side of the augmented RREF, we can read out the solutions as

$$\lambda \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda \in \mathbb{R},$$

- Putting everything together, we obtain the set of all solutions of the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ as

$$\mathcal{S} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{x} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda \in \mathbb{R} \right\} .$$

4.11 Geometry

1. Given $\boldsymbol{u} = \begin{bmatrix} 2 \\ -7 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find the line L passing through the point $\boldsymbol{u}$ in the direction $\boldsymbol{v}$. Is $\boldsymbol{w}$ on this line?

2. Given $\boldsymbol{u} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$. Is $\boldsymbol{u}, \boldsymbol{v}$ perpendicular to each other?

3. Given $\boldsymbol{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 4 \\ -2 \\ -2 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}$. Find the plane P passing through the point $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

4. Given $\boldsymbol{u} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$. Find the projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ and the distance between $\boldsymbol{u}$ and $\boldsymbol{v}$

5. Given $\boldsymbol{u} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} 2 \\ 8 \\ 1 \end{bmatrix}$. Let P be the plane passing through the point $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. Is $\boldsymbol{x} = \begin{bmatrix} 3 \\ 10 \\ 7 \end{bmatrix}$ on this plane?

4.12 Geometry solution

1. Given $\boldsymbol{u} = \begin{bmatrix} 2 \\ -7 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find the line L passing through the point $\boldsymbol{u}$ in the direction $\boldsymbol{v}$. Is $\boldsymbol{w}$ on this line?

$$L = \Big\{ \boldsymbol{u} + t\boldsymbol{v} \ : \ t \in \mathbb{R} \Big\} = \left\{ \begin{bmatrix} 2 \\ -7 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \end{bmatrix} \ : \ t \in \mathbb{R} \right\} = \begin{bmatrix} 2 - 3t \\ -7 \end{bmatrix}, t \in \mathbb{R}$$

To check is $\boldsymbol{w}$ on this line, $\boldsymbol{w} \in L \iff \exists t \in \mathbb{R}$ such that $\begin{bmatrix} 2 - 3t \\ -7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. There is no t that the equality holds (since $-7 \neq 2$), so $\boldsymbol{w} \notin L$

2. Given $\boldsymbol{u} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$. Is $\boldsymbol{u}, \boldsymbol{v}$ perpendicular to each other?

$$\angle(\boldsymbol{u}, \boldsymbol{v}) = \cos^{-1} \frac{\boldsymbol{u}^\top \boldsymbol{v}}{\|\boldsymbol{u}\| \|\boldsymbol{v}\|} = \cos^{-1} \frac{5(3) + (4)(-4) + 1}{\|\boldsymbol{u}\| \|\boldsymbol{v}\|} = \cos^{-1} 0 = 90^\circ \iff \boldsymbol{u} \perp \boldsymbol{v}$$

3. Given $\boldsymbol{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 4 \\ -2 \\ -2 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}$. Find the plane P passing through the point $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

$$P = \Big\{ \boldsymbol{x} : \boldsymbol{n}^\top \boldsymbol{x} = d \Big\} = \left\{ \boldsymbol{x} : \begin{bmatrix} a \\ b \\ c \end{bmatrix}^\top \boldsymbol{x} = d \right\} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ : \ \begin{bmatrix} a \\ b \\ c \end{bmatrix}^\top \begin{bmatrix} x \\ y \\ z \end{bmatrix} = d \right\} = \Big\{ (x, y, z) \ : \ ax + by + cz = d \Big\}$$

$$\begin{array}{rcl} \boldsymbol{u} \in P & \iff & a - 2b + c = d \\ \boldsymbol{v} \in P & \iff & 4a - 2b - 2c = d \\ \boldsymbol{w} \in P & \iff & 4a + b + 4c = d \end{array} \iff \begin{bmatrix} 1 & -2 & 1 & -1 \\ 4 & -2 & -2 & -1 \\ 4 & 1 & 4 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the system gives

$$a = \frac{d}{6}, \qquad b = -\frac{d}{3}, \qquad c = \frac{d}{6}, \qquad d \in \mathbb{R}$$

4. Given $\boldsymbol{u} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$. Find the projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ and the distance between $\boldsymbol{u}$ and $\boldsymbol{v}$

$$\text{proj}_{\boldsymbol{v}}(\boldsymbol{u}) = \frac{\boldsymbol{u}^\top \boldsymbol{v}}{\|\boldsymbol{v}\|^2} \boldsymbol{v} = \frac{3 - 12 + 28}{3^2 + 4^2 + 7^2} \boldsymbol{v} = \frac{19}{74} \boldsymbol{v}, \quad \text{dist}(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{(1-3)^2 + (-3-4)^2 + (4-7)^2} = \sqrt{62}$$

5. Given $\boldsymbol{u} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} 2 \\ 8 \\ 1 \end{bmatrix}$. Let P be the plane passing through the point $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. Is $\boldsymbol{x} = \begin{bmatrix} 3 \\ 10 \\ 7 \end{bmatrix}$ on this plane?

$$\boldsymbol{x} \in P \iff \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} & \boldsymbol{w} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \boldsymbol{x} \text{ has a non-zero solution} \iff \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 8 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 7 \end{bmatrix} \text{ has a non-zero solution}$$

Solving the linear system gives $a = 2, b = 7, c = -3$, then $\boldsymbol{x} = 2\boldsymbol{u} + 7\boldsymbol{v} - 3\boldsymbol{w}$ so $\boldsymbol{x} \in P$

5 Abstract Linear algebra

1. Given two vectors $\boldsymbol{a}, \boldsymbol{b}$, how do you tell are they linearly independent? What mathematical object you will compute to determine that they are/aren't linearly independent?

2. Given a square matrix $\boldsymbol{M}$, how do you tell it is full rank? What mathematical object you will compute to determine that it is/isn't full rank?

3. Given two square matrices $\boldsymbol{X}, \boldsymbol{Y}$ such that $\boldsymbol{X}\boldsymbol{Y} = \boldsymbol{Y}^3\boldsymbol{X}$ and $\boldsymbol{Y}^4 = \boldsymbol{I}$, prove that $\boldsymbol{Y}\boldsymbol{X}\boldsymbol{Y} = \boldsymbol{X}$.

4. Given $\boldsymbol{A} = \begin{bmatrix} 4 & -5 \\ 6 & -9 \end{bmatrix}$. Show that $\boldsymbol{A}^2 + 5\boldsymbol{A} = 6\boldsymbol{I}_2$.

5. Write the vector $\boldsymbol{v} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ as a linear combination of the vectors $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \boldsymbol{u}_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

6. Write the vector $\boldsymbol{v} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$ as a linear combination of the vectors $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix}, \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix}$

5.1 Solution to abstract linear algebra

1. Given two vectors $\boldsymbol{a}, \boldsymbol{b}$, how do you tell are they linearly independent? What mathematical object you will compute to determine that they are/aren't linearly independent?

$$\boldsymbol{a}, \boldsymbol{b} \text{ are linearly independent} \iff \text{for the matrix } \boldsymbol{A} = [\boldsymbol{a}\ \boldsymbol{b}], \text{ the system } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0} \text{ has non-zero solution}$$

2. Given a square matrix $\boldsymbol{M}$, how do you tell it is full rank? What mathematical object you will compute to determine that it is/isn't full rank?

$$\boldsymbol{M} \text{ is full rank} \iff \det \boldsymbol{M} \neq 0$$

3. Given two square matrices $\boldsymbol{X}, \boldsymbol{Y}$ such that $\boldsymbol{X}\boldsymbol{Y} = \boldsymbol{Y}^3\boldsymbol{X}$ and $\boldsymbol{Y}^4 = \boldsymbol{I}$, prove that $\boldsymbol{Y}\boldsymbol{X}\boldsymbol{Y} = \boldsymbol{X}$.

$$\boldsymbol{Y}\boldsymbol{X}\boldsymbol{Y} = \boldsymbol{Y}(\boldsymbol{X}\boldsymbol{Y}) = \boldsymbol{Y}(\boldsymbol{Y}^3\boldsymbol{X}) = (\boldsymbol{Y}\boldsymbol{Y}^3)\boldsymbol{X} = \boldsymbol{Y}^4\boldsymbol{X} = \boldsymbol{I}\boldsymbol{X} = \boldsymbol{X}$$

4. Given $\boldsymbol{A} = \begin{bmatrix} 4 & -5 \\ 6 & -9 \end{bmatrix}$. Show that $\boldsymbol{A}^2 + 5\boldsymbol{A} = 6\boldsymbol{I}_2$.

   Of course you can do the brute force way to show this.
   This equation is true because of Cayley-Hamilton theorem: the charcateristic polynomial of $\boldsymbol{A}$ is

$$p(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \det \begin{bmatrix} \lambda - 4 & 5 \\ -6 & \lambda + 9 \end{bmatrix} = (\lambda - 4)(\lambda + 9) + 30 = \lambda^2 + 5\lambda - 6$$

   Thus by Cayley-Hamilton theorem, $\boldsymbol{A}^2 + 5\boldsymbol{A} - 6\boldsymbol{I}_2 = \boldsymbol{0}$.

5. Write the vector $\boldsymbol{v} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ as a linear combination of the vectors $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \boldsymbol{u}_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$$\boldsymbol{v} \text{ as a linear combination of } \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \iff \boldsymbol{U}\boldsymbol{x} = \boldsymbol{v} \text{ has a non-zero solution for } \boldsymbol{x} \iff \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

   Solving the linear system gives $\boldsymbol{v} = -6\boldsymbol{u}_1 + 3\boldsymbol{u}_2 + 2\boldsymbol{u}_3$

6. Write the vector $\boldsymbol{v} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$ as a linear combination of the vectors $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix}, \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix}$

$$\boldsymbol{v} \text{ as a linear combination of } \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \iff \boldsymbol{U}\boldsymbol{x} = \boldsymbol{v} \text{ has a non-zero solution for } \boldsymbol{x} \iff \begin{bmatrix} 1 & 2 & 2 \\ -3 & -4 & -5 \\ 2 & -1 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

   Solving the linear system we found that it has no solution, so $\boldsymbol{v}$ cannot be written as a linear a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$

End (ver. October 18, 2024)