CO327 Deterministic OR Models (2021-Spring)

# Tutorial 1. Consolidation examples

Formulate problems into linear programs, LP in canonical form and standard form

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# Which is linear optimization (LO) which is not?

1. $\min_{x} 3x^3 + 6x^2 + 9x + 12.$
2. $\max_{x,y}\ x - 2y \ \text{ s.t. }\ x + y = 1, 3x + 4y = 2.$
3. $\min_{x,y}\ x \ \text{ s.t. }\ y = 2.$
4. $\max_{x}\ x \ \text{ s.t. }\ y = \sqrt{-1},\ \pi x < 0.$
5. $\min_{x}\ -x \ \text{ s.t. }\ 3\log x \leq 0.$
6. $\min_{\text{your age}}\ \text{your age} + \text{your mum's age} \ \text{ s.t. }\ \text{your age}^2 \leq 100.$
7. $\max_{x}\ \log x \ \text{ s.t. }\ x \leq 0.$
8. $\max_{y}\ 0 \ \text{ s.t. }\ x + y \leq -3, x \geq y.$
9. $\max_{x}\ 2x + \xi \ \text{ s.t. }\ -3x \leq 1, \xi \sim \mathcal{N}(0,1).$
10. $\max_{\mathbf{x}}\ \mathbf{x} \ \text{ s.t. }\ \mathbf{x} \in \mathbb{R}^n$
11. $\min_{x}\max_{y}\ x - y \ \text{ s.t. }\ x$ is prime and $y$ is the smallest even prime.

1. N. It is polynomial optimization (an advanced course that is after linear programming).
2. Y.
3. Y. $y$ is redundant. This problem has no solution (unbounded).
4. "Y". $y$ and $\pi$ are redundant. This problem has no solution (unbounded due to strict inequality).
5. Y. $\log x \leq 0 \implies 0 < x \leq 1$.
6. Y. $\text{age}^2 \leq 100 \implies -10 \leq \text{age} \leq 10$.
7. N. log is nonlinear
8. Y. $0 = 0y$. Here $x$ is a free variable. This problem is a Constraint Satisfaction Problem.
9. Y. But this is a stochastic linear program.
10. Y. It has no solution ($\mathbf{x}^*$ at infinity) / unbounded LP.
11. N. It is integer programming (precisely integer linear programming). Sol is = 2 - 2 = 0.

# Step-by-step example: butcher shop ... (1/3)

- A butcher shop makes its meat loaf from a combination of beef and pork. The beef contains 80% meat and 20% fat, and costs the shop 80$ per kilo; the pork contains 68% meat and 32% fat, and costs 60$ per kilo. How much of each kind of meat should the shop use in each kilo of meat loaf if it wants to minimize its cost and to keep the fat of the meat loaf to no more than 25%?

- Step 1. Identify the goal.
  Goal: minimize the cost $z$, of meat loaf per kilo.

- Step 2. Identify the variable.
  $x_1$: amount (in % per kilo) of beef, $x_2$: amount of pork.

# Step-by-step example: butcher shop... (2/3)

- Step 3. Formulate the objective function

$$z = f(\mathbf{x}) = 80x_1 + 60x_2.$$

- Step 4. Formulate the primary constraint.
  Each kilo of meat has less than 25% fat

$$0.2x_1 + 0.32x_2 \leq 0.25.$$

- Step 5. Formulate the hidden constraint (avoid improper modeling).
  Amount of beef and amount of pork sum to 1

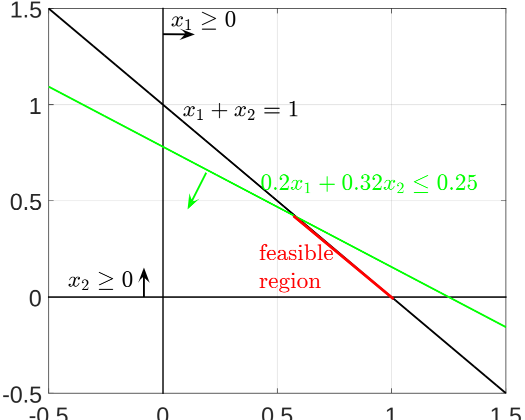
$$x_1 + x_2 = 1.$$

  Amount of beef and amount of pork cannot be negative

$$x_1 \geq 0, \;\; x_2 \geq 0.$$

# Step-by-step example: butcher shop ... (3/3)

$$\min_{x_1,x_2} \quad 80x_1 + 60x_2 \quad \text{s.t.} \quad \begin{aligned} &0.2x_1 + 0.32x_2 \leq 0.25 \\ &x_1 + x_2 = 1 \\ &x_1 \geq 0 \\ &x_2 \geq 0 \end{aligned}$$



# One more example: Cups ... (1/4)

- You produce beer mugs and wine glasses. You sell a beer mugs for 25$ and you sell a wine glass for 20$. To make a beer mug you need 20 resins, while making a wine glass requires 12. You have at most 1800 resins. About 15 of either product can be produced per hour, and you want to work at most 8 hours.

  Formulate this problem as a LP to maximize your profit.

# One more example: Cups ... (2/4)

- Variable: $x_1$ number of beer mug produced; $x_2$: number of wine glasses produced.
- Objective function: $25x_1 + 20x_2$
- Constraint 1 (explicit): resin consumption constraint

$$20x_1 + 12x_2 \leq 1800.$$

- Constrain 2 (explicit): labor constraint

$$\frac{x_1}{15} + \frac{x_2}{15} \leq 8.$$

- Constraint 3 (implicit): you cannot make negative amount of cups

$$x_1 \geq 0, \quad x_2 \geq 0.$$

- Constraint 4 (implicit): you can only sell integer amount of cups

$$x_1 \in \mathbb{Z}, \quad x_2 \in \mathbb{Z}.$$

Note: as $x_i \geq 0$, you can replace $\mathbb{Z}$ by $\mathbb{N}$.

# One more example: Cups ... (3/4)

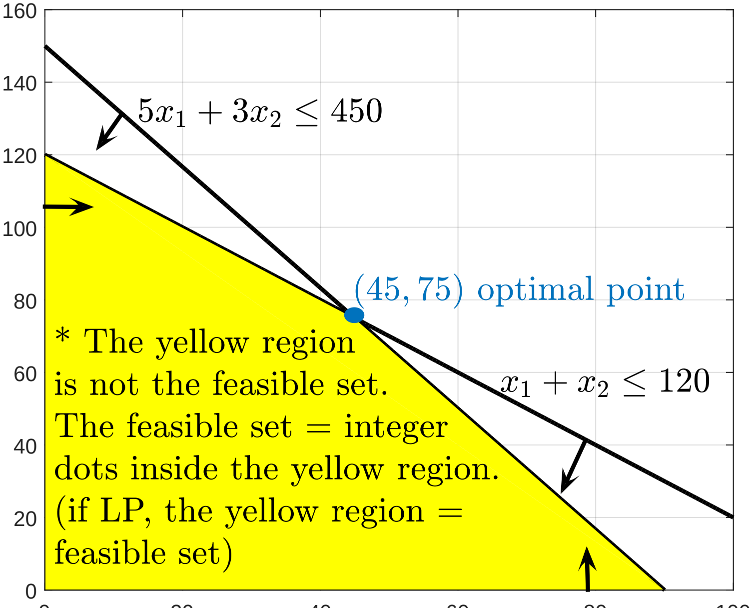
- The IP

$$\max_{x_1,x_2} \quad 25x_1 + 20x_2 \quad \text{s.t.} \quad \begin{aligned} & 20x_1 + 12x_2 \leq 1800 \\ & \frac{x_1}{15} + \frac{x_2}{15} \leq 8 \\ & x_1 \geq 0, \quad x_2 \geq 0 \\ & x_1 \in \mathbb{N}, \quad x_2 \in \mathbb{N} \end{aligned}$$

- Simplify

$$\max_{x_1,x_2} \quad 5(5x_1 + 4x_2) \quad \text{s.t.} \quad \begin{aligned} & 5x_1 + 3x_2 \leq 450 \\ & x_1 + x_2 \leq 120 \\ & x_1 \geq 0, \quad x_2 \geq 0 \\ & x_1 \in \mathbb{N}, \quad x_2 \in \mathbb{N} \end{aligned}$$

# One more example: Cups ... (4/4)



# A preview of a theory and algorithm

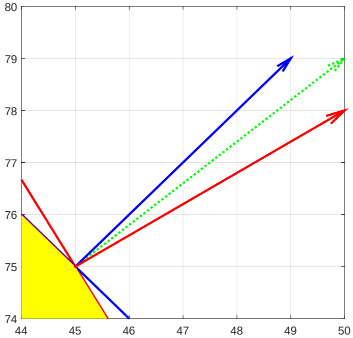
- Consider LP now: ignore $x_1 \in \mathbb{N}, \ x_2 \in \mathbb{N}$, so yellow region $=$ feasible set.
- The yellow region is a bounded polygon.
- Fact: optimal sol. is at the corner (vertex/extreme point) of this polygon.
- Theorem: optimal sol. at extreme points (discuss in theory section.).
- Dantzig's simplex algorithm: search among the vertices.

# A preview of geometry and duality

$$\max_{x_1,x_2} \quad 5 \begin{bmatrix} 5 \\ 4 \end{bmatrix}^\top \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix} 5 & 3 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 450 \\ 120 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 \in \mathbb{N}, \quad x_2 \in \mathbb{N}$$




- Fact: green vector $\in$ cone generated by red and blue vectors.
- Fact: $\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} 0.5 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} 2.5.$
- This corresponds to $\mathbf{c} = \mathbf{A}^\top \mathbf{y}$, $\mathbf{y} \geq 0$.
- This is Lagrangian duality.

# Optimal sol. at the corner only?

$$
\begin{aligned}
\max_{x_1,x_2} \quad & \begin{bmatrix}1\\0\end{bmatrix}^\top \begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1 \\
\text{s.t.} \quad & x_1 \geq 0 \\
& x_2 \geq 0 \\
& x_1 \leq 1 \\
& x_2 \leq 1
\end{aligned}
$$

- Optimizer is $\begin{bmatrix}1\\t\end{bmatrix}$, $t \in [0,1] \implies$ infinitely many solutions.
- Optimizer is at the right edge of the box, not just the corner.

# LP modeling: recap

- Step 1. Identify the variable.
- Step 2. Identify the goal and formulate the objective function
- Step 3. Formulate the explicit constraint(s).
- Step 4. Formulate the hidden/implicit constraint(s).
- Important notes: you can always add more variables in the modeling. Sometimes it is in fact better to have more variables: they give you more freedom.

# Expressing LP in canonical form and standard form

- Canonical form and standard form

$$\max_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x} \qquad \mathbf{A}\mathbf{x} \le \mathbf{b} \qquad\qquad \max_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x} \qquad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \ge 0$$

  where the inequality are applied elementwise.

- Some authors consider $\min_{\mathbf{x}} \; \mathbf{c}^\top \mathbf{x}$, $\mathbf{A}\mathbf{x} \le \mathbf{b}$, $\mathbf{x} \ge 0$ as canonical form, however $\mathbf{x} \ge 0$ can be absorbed into $\mathbf{A}\mathbf{x} \le \mathbf{b}$ as $\begin{bmatrix} \mathbf{A} \\ -\mathbf{I} \end{bmatrix} \mathbf{x} \le \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}$.

- All LP can be transformed to canonical form and standard form.

# Example of transform to canonical form ... (1/2)

- Express the LP in canonical form

$$\min_{x_1,x_2} \quad 80x_1 + 60x_2 \quad \text{s.t.} \quad \begin{array}{lcl} 0.2x_1 + 0.32x_2 & \leq & 0.25 \\ x_1 + x_2 & = & 1 \\ x_1 & \geq & 0 \\ x_2 & \geq & 0 \end{array}$$

- Objective function

$$\min_{x_1,x_2} 80x_1 + 60x_2 = \max_{x_1,x_2} \begin{bmatrix} -80 \\ -60 \end{bmatrix}^\top \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{c}^\top \mathbf{x}.$$

- Constraint set

$$\begin{array}{rcl} 0.2x_1 + 0.32x_2 & \leq & 0.25 \\ x_1 & \geq & 0 \\ x_2 & \geq & 0 \end{array} \quad \Longrightarrow \quad \begin{array}{rcl} 0.2x_1 + 0.32x_2 & \leq & 0.25 \\ -x_1 & \leq & 0 \\ -x_2 & \leq & 0 \end{array}$$

- What about the equality constraint $x_1 + x_2 = 1$?

# Example of transform to canonical form ... (2/2)

- Equality becomes two inequalities.

$$x_1 + x_2 = 1 \quad \Longrightarrow \quad \begin{array}{rcl} x_1 + x_2 & \geq & 1 \\ x_1 + x_2 & \leq & 1 \end{array} \quad \Longrightarrow \quad \begin{array}{rcl} -x_1 - x_2 & \leq & -1 \\ x_1 + x_2 & \leq & 1 \end{array}$$

- Now the LP

$$\max_{x_1, x_2} \quad -80x_1 - 60x_2 \quad \text{s.t.} \quad \begin{array}{lcl} 0.2x_1 + 0.32x_2 & \leq & 0.25 \\ -x_1 - x_2 & \leq & -1 \\ x_1 + x_2 & \leq & 1 \\ -x_1 & \leq & 0 \\ -x_2 & \leq & 0 \end{array}.$$

In matrix-vector notation

$$\mathbf{c} = \begin{bmatrix} -80 \\ -60 \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} 0.2 & 0.32 \\ -1 & -1 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 0.25 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

# Practice problem

- Transform the LP to canonical form

$$\begin{aligned} \min_{x_1, x_2} \quad & 2x_1 - x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 2 \\ & 3x_1 + 2x_2 \leq 4 \\ & x_1 + 2x_2 = 3 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

- Answer:

$$\mathbf{c} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} -1 & -1 \\ 3 & 2 \\ -1 & -2 \\ 1 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ 4 \\ -3 \\ 3 \\ 0 \\ 0 \end{bmatrix}.$$

# Expressing LP in standard form ... (1/3)

- The LP from the Cup example (ignoring the integer constraints) is already in canonical form.

$$\max_{x_1,x_2} \ 5\begin{bmatrix}5\\4\end{bmatrix}^\top \begin{bmatrix}x_1\\x_2\end{bmatrix} \quad \text{s.t.} \quad \begin{bmatrix}5 & 3\\1 & 1\\-1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} \leq \begin{bmatrix}450\\120\\0\\0\end{bmatrix}.$$

Transform this to standard form.

# Expressing LP in standard form ... (2/3)

- How to transform inequality constraint like $5x_1 + 3x_2 \leq 450$ into equality constraint? Recall few slides ago:

  **you can always add more variables in the modeling.**

  So

  $$5x_1 + 3x_2 \leq 450 \implies \begin{array}{rcl} 5x_1 + 3x_2 + s_1 & = & 450 \\ s_1 & \geq & 0 \end{array}$$

  the newly added $s_1$ is called slack variable.

- Similarly,

  $$5x_1 + 3x_2 \geq 450 \implies \begin{array}{rcl} 5x_1 + 3x_2 - s_1 & = & 450 \\ s_1 & \geq & 0 \end{array}$$

- Due to the definition of standard form, you want the newly added variable to be nonnegative.

# Expressing LP in standard form ... (3/3)

- The original Cup LP

$$\max_{x_1,x_2} \; 5\begin{bmatrix}5\\4\end{bmatrix}^\top \begin{bmatrix}x_1\\x_2\end{bmatrix} \quad \text{s.t.} \quad \begin{bmatrix}5 & 3\\1 & 1\\-1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} \leq \begin{bmatrix}450\\120\\0\\0\end{bmatrix}.$$

- In standard form

$$\max_{x_1,x_2,s_1,s_2} \; 5\begin{bmatrix}5\\4\\0\\0\end{bmatrix}^\top \begin{bmatrix}x_1\\x_2\\s_1\\s_2\end{bmatrix} \text{ s.t. } \begin{bmatrix}5 & 3 & 1 & 0\\1 & 1 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\s_1\\s_2\end{bmatrix} = \begin{bmatrix}450\\120\end{bmatrix}, \begin{bmatrix}x_1\\x_2\\s_1\\s_2\end{bmatrix} \geq 0.$$

# Interval bounds

- $|x_i| \leq b_i$ is equivalent to $-b_i \leq x_i \leq b_i$.
- In general, interval bound has the form $l_i \leq x_i \leq u_i$.
- To turn interval bound to standard form, again you add slack variable. Again, due to the definition of standard form, you want the newly added variable to be nonnegative:

$$
\begin{aligned}
& l_i \leq x_i \leq u_i \\
\overset{x_i = l_i + s_i}{\iff} \quad & l_i \leq l_i + s_i \leq u_i \\
\iff \quad & 0 \leq s_i \leq u_i - l_i
\end{aligned}
$$

- Example

$$
\begin{aligned}
\max_x \quad & x \\
\text{s.t.} \quad & 2019 \leq x \leq 2020
\end{aligned}
\implies
\begin{aligned}
\max_s \quad & 2019 + s \\
\text{s.t.} \quad & s \leq 1 \\
& s \geq 0
\end{aligned}
$$

# Variable without bound / free variable

- Example

$$\max_{x_1, x_2} \quad x_1 + x_2$$
$$\text{s.t.} \quad x_1 \geq 0$$

- To transform this LP to standard form, by definition we must have $\mathbf{x} \geq 0$ in the constraint.

- To do so we transform $x_2 = s_1 - s_2$ with $s_1 \geq 0$ and $s_2 \geq 0$.

- In standard form

$$\max_{x_1, s_1, s_2} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}^\top \begin{bmatrix} x_1 \\ s_1 \\ s_2 \end{bmatrix} \text{ s.t. } \begin{bmatrix} x_1 \\ s_1 \\ s_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

# Everything together ... (1/7)

- Transform the LP to standard form

$$
\begin{aligned}
\min_{x_1,x_2,x_3,x_4} \quad & 2x_1 - x_2 + x_3 + x_4 \\
\text{s.t.} \quad & x_1 + x_2 \geq 2 \\
& 3x_1 + 2x_2 \leq 4 \\
& x_1 + 2x_2 = 3 \\
& 2 \leq x_1 \leq 5 \\
& x_2 \geq 0 \\
& x_4 = 0
\end{aligned}
$$

# Everything together ... (2/7)

- $x_4$ is redundant

$$\begin{aligned} \min_{x_1,x_2,x_3,x_4} \quad & 2x_1 - x_2 + x_3 + x_4 \\ \text{s.t.} \quad & x_1 + x_2 \geq 2 \\ & 3x_1 + 2x_2 \leq 4 \\ & x_1 + 2x_2 = 3 \\ & 2 \leq x_1 \leq 5 \\ & x_2 \geq 0 \\ & x_4 = 0 \end{aligned}$$

Becomes

$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & 2x_1 - x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 \geq 2 \\ & 3x_1 + 2x_2 \leq 4 \\ & x_1 + 2x_2 = 3 \\ & 2 \leq x_1 \leq 5 \\ & x_2 \geq 0 \end{aligned}$$

# Everything together ... (3/7)

- Rewrite the objective function

$$\begin{aligned}
\min_{x_1,x_2,x_3} \quad & 2x_1 - x_2 + x_3 \\
\text{s.t.} \quad & x_1 + x_2 \geq 2 \\
& 3x_1 + 2x_2 \leq 4 \\
& x_1 + 2x_2 = 3 \\
& 2 \leq x_1 \leq 5 \\
& x_2 \geq 0
\end{aligned}$$

Becomes

$$\begin{aligned}
\max_{x_1,x_2,x_3} \quad & -2x_1 + x_2 - x_3 \\
\text{s.t.} \quad & x_1 + x_2 \geq 2 \\
& 3x_1 + 2x_2 \leq 4 \\
& x_1 + 2x_2 = 3 \\
& 2 \leq x_1 \leq 5 \\
& x_2 \geq 0
\end{aligned}$$

# Everything together ... (4/7)

- Rewrite $x_1$ as $2 + s_1$

$$\begin{aligned} \max_{x_1, x_2, x_3} \quad & -2x_1 + x_2 - x_3 \\ \text{s.t.} \quad & x_1 + x_2 \geq 2 \\ & 3x_1 + 2x_2 \leq 4 \\ & x_1 + 2x_2 = 3 \\ & 2 \leq x_1 \leq 5 \\ & x_2 \geq 0 \end{aligned}$$

Becomes

$$\begin{aligned} \max_{s_1, x_2, x_3} \quad & -4 - 2s_1 + x_2 - x_3 \\ \text{s.t.} \quad & 2 + s_1 + x_2 \geq 2 \\ & 6 + 3s_1 + 2x_2 \leq 4 \\ & 2 + s_1 + 2x_2 = 3 \\ & 0 \leq s_1 \leq 3 \\ & x_2 \geq 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \max_{s_1, x_2, x_3} \quad & -4 - 2s_1 + x_2 - x_3 \\ \text{s.t.} \quad & s_1 + x_2 \geq 0 \\ & 3s_1 + 2x_2 \leq -2 \\ & s_1 + 2x_2 = 1 \\ & s_1 \leq 3 \\ & s_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

# Everything together ... (5/7)

- Rewrite $s_1 + 2x_2 = 1$ as two inequalities

$$\max_{s_1,x_2,x_3} \quad -4 - 2s_1 + x_2 - x_3 \quad \text{s.t.} \quad \begin{aligned} & s_1 + x_2 \geq 0 \\ & 3s_1 + 2x_2 \leq -2 \\ & s_1 + 2x_2 = 1 \\ & s_1 \leq 3 \\ & s_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

Becomes

$$\max_{s_1,x_2,x_3} \quad -4 - 2s_1 + x_2 - x_3 \quad \text{s.t.} \quad \begin{aligned} & s_1 + x_2 \geq 0 \\ & 3s_1 + 2x_2 \leq -2 \\ & s_1 + 2x_2 \geq 1 \\ & s_1 + 2x_2 \leq 1 \\ & s_1 \leq 3 \\ & s_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

# Everything together ... (6/7)

- Convert "non-standard" ineqs. to eqs. by adding slack variables

$$\max_{s_1,x_2,x_3} \quad -4-2s_1+x_2-x_3 \quad \text{s.t.} \quad \begin{aligned} & s_1+x_2 \geq 0 \\ & 3s_1+2x_2 \leq -2 \\ & s_1+2x_2 \geq 1 \\ & s_1+2x_2 \leq 1 \\ & s_1 \leq 3 \\ & s_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

Becomes

$$\max_{s_1,\cdots,s_6,x_2,x_3} \quad -4-2s_1+x_2-x_3 \quad \text{s.t.} \quad \begin{aligned} & s_1+x_2-s_2=0 \\ & 3s_1+2x_2+s_3=-2 \\ & s_1+2x_2-s_4=1 \\ & s_1+2x_2+s_5=1 \\ & s_1+s_6=3 \\ & s_1,\cdots,s_6,x_2 \geq 0 \end{aligned}$$

# Everything together ... (7/7)

- Convert the free variable $x_3 = u_1 - v_1$.

$$\max_{s_1,\cdots,s_6,x_2,x_3} \quad -4 - 2s_1 + x_2 - x_3 \quad \text{s.t.} \quad \begin{aligned} & s_1 + x_2 - s_2 = 0 \\ & 3s_1 + 2x_2 + s_3 = -2 \\ & s_1 + 2x_2 - s_4 = 1 \\ & s_1 + 2x_2 + s_5 = 1 \\ & s_1 + s_6 = 3 \\ & s_1, \cdots, s_6, x_2 \geq 0 \end{aligned}$$

Becomes

$$\begin{aligned} \max_{s_1,\cdots,s_6,x_2,u_1,v_1} \quad & -4 - 2s_1 + x_2 - u_1 + v_1 \\ \text{s.t.} \quad & s_1 + x_2 - s_2 = 0 \\ & 3s_1 + 2x_2 + s_3 = -2 \\ & s_1 + 2x_2 - s_4 = 1 \\ & s_1 + 2x_2 + s_5 = 1 \\ & s_1 + s_6 = 3 \\ & s_1, \cdots, s_6, x_2, u_1, v_1 \geq 0 \end{aligned}$$

# The two LPs

- Original LP in $\mathbb{R}^4$ ($\mathbb{R}^3$)

$$\begin{aligned}
\min_{x_1,x_2,x_3,x_4} \quad & 2x_1 - x_2 + x_3 + x_4 \\
\text{s.t.} \quad & x_1 + x_2 \geq 2 \\
& 3x_1 + 2x_2 \leq 4 \\
& x_1 + 2x_2 = 3 \\
& 2 \leq x_1 \leq 5 \\
& x_2 \geq 0 \\
& x_4 = 0
\end{aligned}$$

- Standard form in $\mathbb{R}^9$

$$\begin{aligned}
\max_{s_1,\cdots,s_6,x_2,u_1,v_1} \quad & -4 - 2s_1 + x_2 - u_1 + v_1 \\
\text{s.t.} \quad & s_1 + x_2 - s_2 = 0 \\
& 3s_1 + 2x_2 + s_3 = -2 \\
& s_1 + 2x_2 - s_4 = 1 \\
& s_1 + 2x_2 + s_5 = 1 \\
& s_1 + s_6 = 3 \\
& s_1, \cdots, s_6, x_2, u_1, v_1 \geq 0
\end{aligned}$$

# Summary of expressing LP in canonical or standard form

- Always maximization problem.
  To transform a min. problem to a max. problem, multiply the objective function by -1.

- $\mathbf{a}^\top \mathbf{x} = b$ becomes $\mathbf{a}^\top \mathbf{x} \geq b$ and $\mathbf{a}^\top \mathbf{x} \leq b$.

- $\mathbf{a}^\top \mathbf{x} \leq b$ becomes $\begin{bmatrix} \mathbf{a} \\ 1 \end{bmatrix}^\top \begin{bmatrix} \mathbf{x} \\ s_1 \end{bmatrix} = b$ and $s_1 \geq 0$. Similar for $\mathbf{a}^\top \mathbf{x} \geq b$ case.

- $l \leq x \leq u$ becomes $0 \leq s \leq u - l$.

- Free variable has to be presented in the inequality $\mathbf{x} \geq 0$ by writing the free variable as difference of two nonnegative variables.

# Now what

- Formulate problems into linear programs.
- LP in canonical form and standard form

Next: lecture 2.

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