CO327 Deterministic OR Models (2021-Spring)

2. Introduction to Linear Integer Program

What is IP, Formulate problems into IP, Example of IP: Tower defense

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What is LIP

- LP in standard form

$$\begin{aligned} \max_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

- LIP

$$\begin{aligned} \max_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \\ & \mathbf{x} \in \mathbb{N}^n \end{aligned}$$

- Note: $\mathbf{x} \in \mathbb{Z}^n$ and $\mathbf{x} \geq 0$ gives $\mathbf{x} \in \mathbb{N}^n$.

Why integer?

- Representation of "Yes" and "No"

$$\text{Yes} = 1 \qquad \text{No} = 0.$$

- **Binary thinking**
  - You may wonder why "No" $=0$? Why not No $=-1$?
  - This means you are thinking the variable in $\mathbb{R}$ in *denary number system*: real numbers can be used to express measurements of continuous quantities.
  - Here you should think using *Binary number system* / *Boolean algebra*, i.e., the arithmetic of $1+1=0$, $0+0=0$.
  - So we use integer $\{1,0\}$ to represent yes-no.

Different kinds of programs

- We focus on linear program in this course, so the “L” can be dropped.
- Binary IP (‘beep’)
  - All variable are either 0 or 1.
  - Correspond to binary decision.
- Pure IP
  - All variable are in $\mathbb{N}$.
  - Corresponds to categorical / (finite)-multiple decision.
- Pure continuous LP
  - All variable are in $\mathbb{R}$.
  - Correspond to continuous decision.
  - For real number in the interval $[0, 1]$, you can view it as *fuzzy decision*[1].
- Mixed IP (‘Meep’)
  - Some variables are integers, some are not.
  - A mix between continuous and discrete optimization.

[1] Then we have Fuzzy linear programming.

Example of BIP

- Example. Tourism planning.

Decision	Y/N	Decision variable	Cost	Joy
1	Visit Belgium	x_B	30	5
2	Visit Netherlands	x_N	100	20
3	Visit Luxembourg	x_L	70	7

- Maximize joy while limiting budget below 100.

$$\begin{aligned} \max_{x_B, x_N, x_L} \quad & 5x_B + 20x_N + 7x_L \\ \text{s.t.} \quad & 30x_B + 100x_N + 70x_L \leq 100 \\ & x_B \in \{0, 1\} \\ & x_N \in \{0, 1\} \\ & x_L \in \{0, 1\} \end{aligned}$$

where $1 =$ visit and $0 =$ not visit.

Mathematical modeling of binary decision

- If you visit Netherlands, you don't have money for other countries.
  $\implies$ *mutually exclusive variables*

$$x_B + x_N \leq 1, \;\; x_L + x_N \leq 1.$$

- You prefer Belgium over Luxembourg (*contingent decision*)

$$x_L \leq x_B.$$

- Visit at least 2 countries among the 3 destinations.

$$x_B + x_N + x_L \geq 2.$$

- You have to visit either Belgium or Netherlands

$$x_B + x_N = 1.$$

- You either visit both Belgium and Netherlands, or don't visit both.

$$x_B - x_N = 0.$$

Mixed-Integer programming MIP

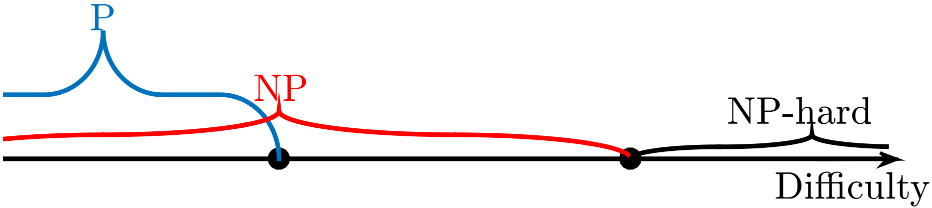
$$
\begin{aligned}
\max_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\
\text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\
& \mathbf{x} \geq 0 \\
& \text{some } x_i \in \mathbb{N}
\end{aligned}
$$

- MIP $=$ continuous LP $\cap$ discrete IP.
- MIP itself is a BIG[2] research.

[2] "BIG" means big and "business-industry-government".

BIP is hard

- Exponential growth of complexity: 1 variable $x \in \{0, 1\}$, 2 choices.
  Two variables $x_1 \in \{0, 1\}, x_2 \in \{0, 1\}$ gives 4 choices
  $$00, \quad 10, \quad 01, \quad 11,$$
  and n variables gives 2^n choices $\implies$ exponential growth of number of choices.
- Difficult to find (global) optimal solution. In fact, BIP is NP-hard.
- In comparison, LP is in P (polynomial solvable, by e.g. ellipsoid method, interior point method).
- WTF are P, NP, NP-hard:



Methods to solve IP

- LP-relaxation: may not work.

- Brute-force: enumerate all possible cases.
  - 100% work
  - Only works efficiently for small problem.
  - Run-time for big problem can over the heat death of the universe.

- Deterministic method
  - Branch-and-Bound
  - Cutting-plane method
  - Branch-and-cut (if time permits)

  ... basically just a smarter brute-force.

- Probabilistic heuristics (not this course)
  - Simulated Annealing
  - Genetic Algorithm
  - Ant Colony / particle swarm

  ... basically just gambling.

*CO327 focuses on models $\implies$ not focus on methods.

An example of real-life problem that is IP: Tower defence

Figure: Kingdom Rush, a tower defense game. Source: internet.

Actually, when you place a tower, your brain is performing a decision making process on fuzzily solving a binary integer optimization problem!

What is tower defence

Figure: A simple tower defence game. Source: internet.

- Goal: place turret that destroys the enemies, preventing them from reaching the destination.

Tower defence is an integer program

- You construct a single type of turret and place it on the map to attack enemies.
- Decision variable: where to place the turret.

$$x_{ij} = 1 \quad \text{place turret at location } (i, j)$$
$$x_{ij} = 0 \quad \text{do not place turret at location } (i, j)$$

  Or you can combine the indices into a single index J (see next slide).

- What affects the decision making
  - Range of the turret.
  - The geographical information.
  - Problem specific constraint(s).
- In fact, Tower defence $\in$ *Set Cover problems*.

Set cover problem

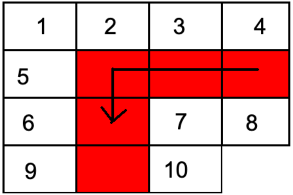
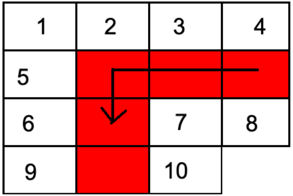


Figure: A simple example. The red blocks is the path of the enemy.

- Decision variables for this example: $x_1, x_2, \ldots, x_{10}$, here we use single indexing (not (i, j)).
- Game rule / information that affect the decision making:
  - Turrets can only hit all the 4 adjacent blocks (but they hit all the 4 adjacent block simultaneously).
  - You want to make sure all the red block is targeted by at least TWO turrets.
- Objective: you want to use the least number of turrets.
- Formulate this problem as a BIP.

Formulate the cost

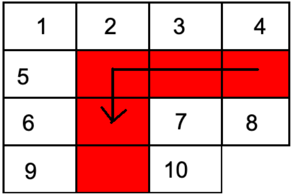


- The cost function: number of turret placed: $\sum_{i=1}^{10} x_i = \mathbf{1}^\top \mathbf{x}$
- BIP: minimize the cost subject to constraint

$$\min_{\mathbf{x}} \mathbf{1}^\top \mathbf{x} \ \text{ s.t. } \ \mathbf{x} \in \mathcal{C} \cap \mathbf{x} \in \{0,1\}^{10},$$

  where $\mathcal{C}$ is the constraint set to be determined.
- We can see $x_1 = 0$: placing a turret there is useless.

Formulating the constraint



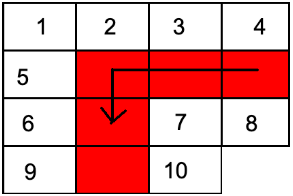
- Each red block is targeted by at least 2 turrets

$$
\begin{array}{lllllllllll}
x_2 & & & +x_5 & & & & & & \geq 2 \\
& x_3 & & & & +x_7 & & & & \geq 2 \\
& & x_4 & & & & +x_8 & & & \geq 2 \\
& & & & x_6 & +x_7 & & & & \geq 2 \\
& & & & & & & x_9 & +x_{10} & \geq 2
\end{array}
$$

This system of inequalities is the set $\mathcal{C}$.

- Based on the constraints, it happened that for this particular BIP, all x_i (except x_1) $=1$.

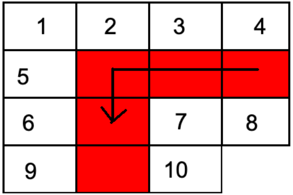
The same problem, with a twist ... (1/2)



- Now turrets can hit the 8 adjacent blocks surrounding it.
- The system of inequalities becomes

$$
\begin{array}{cccccccccccc}
x_1 & +x_2 & +x_3 & & +x_5 & +x_6 & +x_7 & & & & \geq 2 \\
& x_2 & +x_3 & +x_4 & & & +x_7 & +x_8 & & & \geq 2 \\
& & x_3 & +x_4 & & & +x_7 & +x_8 & & & \geq 2 \\
& & & & x_5 & +x_6 & +x_7 & & +x_9 & +x_{10} & \geq 2 \\
& & & & & x_6 & +x_7 & & +x_9 & +x_{10} & \geq 2
\end{array}
$$

The same problem, with a twist ... (2/2)

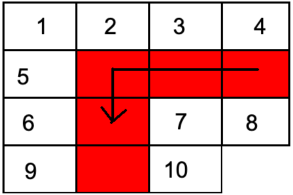


- Now the BIP is

$$
\begin{array}{l}
\min_{\mathbf{x}\in\{0,1\}^{10}} \mathbf{1}^\top \mathbf{x} \text{ s.t.} \\
\begin{array}{cccccccccccc}
x_1 & +x_2 & +x_3 & & +x_5 & +x_6 & +x_7 & & & & \geq 2 \\
& x_2 & +x_3 & +x_4 & & & +x_7 & +x_8 & & & \geq 2 \\
& & x_3 & +x_4 & & & +x_7 & +x_8 & & & \geq 2 \\
& & & & x_5 & +x_6 & +x_7 & & +x_9 & +x_{10} & \geq 2 \\
& & & & & x_6 & +x_7 & & +x_9 & +x_{10} & \geq 2
\end{array}
\end{array}
$$

- What is the optimal solution? Is it unique? If it is not unique, list all the optimal solution. (Assignment)

The same problem, more twists



- Turrets cannot be adjacent to each other.
- Turret at location 7 can only hit the target on its left.
- The boxes have different cost for placing turrets.
- Turrets can only choose to hit one of the block among its 4 adjacent blocks.
- Beside turrets, you can place rocket launcher.
  - Formulate the problem as IP.
  - Formulate the problem as BIP.

Related problem

- Set covering: "cover each region at least once"

$$\min_{\mathbf{x}} \mathbf{c}^\top \mathbf{x} \ \text{ s.t. } \ \mathbf{A}\mathbf{x} \geq \mathbf{1}, \mathbf{x} \in \{0,1\}^m$$

- Set packing: "cover as many region without overlapping"

$$\max_{\mathbf{x}} \mathbf{c}^\top \mathbf{x} \ \text{ s.t. } \ \mathbf{A}\mathbf{x} \leq \mathbf{1}, \mathbf{x} \in \{0,1\}^m$$

- Set partitioning: "cover exactly one"

$$\max_{\mathbf{x}} \mathbf{c}^\top \mathbf{x} \ \text{ s.t. } \ \mathbf{A}\mathbf{x} = \mathbf{1}, \mathbf{x} \in \{0,1\}^m$$

- Things in common: $\mathbf{A} \in \{0,1\}^{n \times m}$

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Applications of Set Covering, Set Packing and Set Partitioning Models: A Survey

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Abstract

Set covering, set packing and set partitioning models are a special class of linear integer programs. These models and their variants have been used to formulate a variety of practical problems in such areas as capital budgeting, crew scheduling, cutting stock, facilities location, graphs and networks, manufacturing, personnel scheduling, vehicle routing and timetable scheduling among others. Based on the special structure of these models, efficient

... more in assignment 2, or future assignments.

Summary

- What is IP.
- Formulate problems into IP / BIP.
- Example of application: Tower defence as a set covering problem.

Next : go to tutorial 2.

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