CO327 Deterministic OR Models (2021-Spring)

2-player zero-sum game

Instructor: Andersen Ang

Combinatorics and Optimization, U.Waterloo, Canada

msxang@uwaterloo.ca, where $\mathbf{x} = \lfloor \pi \rfloor$

Homepage: angms.science

First draft: Janurary 2, 2020 Last update: July 17, 2021

Brief introduction to game theory

- Decision theory: study of selecting optimal actions, the actions do not influence the behaviour of other people around.
- Game theory: study of selecting a good *strategy of actions*, that the actions **do** influence the behaviour of other people around.
- Strategy: a complete way to play the game regardless of what other player does.
- We only focus on 2-player game. Let P1= player1 and P2= player2.

A simple 2-player game: cake division ... (1/3)

- The two players have one cake and they want to divide the cake into two pieces.
- If the division is fair, each player get half of the cake.
- Now they play the cake division game: P1 (the cutter) will cut the cake first. P2 (the chooser) then choose a portion of the cake, and P1 get the remaining part.
- The *payoff* matrix is then

		P2 Choose	
		Pick the large one	Pick the small one
P1 Cut	Even	half,half	half,half
	Uneven	small piece, large piece	large piece, small piece

- Using numbers, the payoff matrix is then

		P2 Choose	
		Pick the large one	Pick the small one
P1 Cut	Even	50,50	50,50
	Uneven	20,80	80,20

A simple 2-player game: cake division ... (2/3)

		P2 Choose	
		Pick the large one	Pick the small one
P1 Cut	Even	50,50	50,50
	Uneven	20,80	80,20

- The $(80, 20)$ means P1 get 80% of the cake, if

  he cut unevenly and then P2 pcik the small one.

  and player2 get 20% of the cake.

- Since we assume players are *rational actor*, they will pick the optimal choice. P2 will not pick the small one.

A simple 2-player game: cake division ... (3/3)

- If they do not play the game and divide the cake fairly, each of them will get 50,50. So any deviation from (50,50) makes the game unfair.
- When the outcome is unfair, one player gains and the other player losses.
- Zero-sum: the gain from a player comes from the loss of another player.
- We can then transform the payoff matrix as

		P2 Choose	
		Pick the large one	Pick the small one
P1 Cut	Even	0	0
	Uneven	-30	30

- The number here refer to how much P1 gains from P2.

A 3-option 2-player zero-sum game

- Consider a 3-option 2-player game: each player can choose between 3 options: A, B and C. The payoff matrix for P1 is

	P2 A	P2 B	P2 C
P1 A	3	1	-1
P1 B	-2	3	2
P1 C	1	-2	4

- The "4" means, if P1 choose C and P2 choose C, then P2 has to pay P1 4$.
- So P1 want to make the choice (based on P2) that maximizes the payoff value.
- So P2 want to make the choice (based on P1) that minimizes the payoff value.
- How do they make choice?

Probabilistic strategy

	P2 A	P2 B	P2 C
P1 A	3	1	-1
P1 B	-2	3	2
P1 C	1	-2	4

- Let x_A, x_B, x_C be the probability of P1 choosing action A,B and C, respectively.
- Because they are probabilities, then naturally we have the constraints

$$0 \leq x_A \leq 1, \quad 0 \leq x_B \leq 1, \quad 0 \leq x_C \leq 1, \quad x_A + x_B + x_C = 1.$$

- Optimal strategy: P1 want to assign the probabilities such that his payoff, in expectation, is maximized regardless of what P2 choose.

Expected payoff of P1

	P2 A	P2 B	P2 C
P1 A	3	1	-1
P1 B	-2	3	2
P1 C	1	-2	4

- The expected payoff of P1 if P2 choose A is

$$p_1 = \text{Payoff}(\text{P1} \mid \text{P2 A}) = 3x_A - 2x_B + x_C$$

- The expected payoff of P1 if P2 choose B is

$$p_2 = \text{Payoff}(\text{P1} \mid \text{P2 B}) = x_A + 3x_B - 2x_C$$

- The expected payoff of P1 if P2 choose C is

$$p_3 = \text{Payoff}(\text{P1} \mid \text{P2 C}) = -x_A + 2x_B + 4x_C$$

- P1 wants all these 3 values as high as possible: P1 want to maximize $\min\{p_1, p_2, p_3\}$

Maximizing the payoff of P1

	P2 A	P2 B	P2 C
P1 A	3	1	-1
P1 B	-2	3	2
P1 C	1	-2	4

- P1 want to maximize $\min\{p_1, p_2, p_3\}$: using the extreme-case modelling trick, introduce a 4th variable x that act as the lower bound for all p_i:

$$\begin{aligned} p_1 = & \quad 3x_A - 2x_B + x_C & \geq & \quad x \\ p_2 = & \quad x_A + 3x_B - 2x_C & \geq & \quad x \\ p_3 = & \quad -x_A + 2x_B + 4x_C & \geq & \quad x \end{aligned}$$

and then push this x as large as you can!

LP for P1

- Finally, the LP of P1's optimal strategy reads

$$\begin{aligned}
\max_{x_A, x_B, x_C, x} \quad & x \\
\text{s.t.} \quad & x_A + x_B + x_C = 1 \\
& x_A \geq 0 \\
& x_B \geq 0 \\
& x_C \geq 0 \\
& 3x_A - 2x_B + x_C \geq x \\
& x_A + 3x_B - 2x_C \geq x \\
& -x_A + 2x_B + 4x_C \geq x
\end{aligned}$$

- Solving this LP yields

$$x_A = \frac{9}{19}, \quad x_B = \frac{6}{19}, \quad x_C = \frac{4}{19}, \quad x = 1$$

which means if P1 choose A with probability $\frac{9}{19}$, B with prob. $\frac{5}{19}$ and C with prob. $\frac{4}{19}$, then regardless of what P2 choose, P1 will gain at least $x = 1$ in expectation.

On P2

	P2 A	P2 B	P2 C
P1 A	3	1	-1
P1 B	-2	3	2
P1 C	1	-2	4

- Let y_A, y_B, y_C be the probability of P2 choosing action A,B and C, respectively.
- Because they are probabilities, then naturally we have the constraints

$$0 \leq y_A \leq 1, \quad 0 \leq y_B \leq 1, \quad 0 \leq y_C \leq 1, \quad y_A + y_B + y_C = 1.$$

- Optimal strategy: P2 want to assign the probabilities such that P1's payoff, in expectation, is **minimized** regardless of what P1 choose.

Expected payoff of P1 under P2

	P2 A	P2 B	P2 C
P1 A	3	1	-1
P1 B	-2	3	2
P1 C	1	-2	4

- The expected payoff of P1 under P2

$$\begin{aligned}
\text{Payoff}\Big(\text{P1} \;\Big|\; \text{P1 A} \big| \text{P2}(y_A, y_B, y_C)\Big) &= 3y_A + y_B - y_C \\
\text{Payoff}\Big(\text{P1} \;\Big|\; \text{P1 B} \big| \text{P2}(y_A, y_B, y_C)\Big) &= -2y_A + 3y_B + 2y_C \\
\text{Payoff}\Big(\text{P1} \;\Big|\; \text{P1 C} \big| \text{P2}(y_A, y_B, y_C)\Big) &= y_A - 2y_B + 4y_C
\end{aligned}$$

- P2 wants all these 3 values to be as low as possible:

$$\begin{aligned}
3y_A + y_B - y_C &\leq y \\
-2y_A + 3y_B + 2y_C &\leq y \\
y_A - 2y_B + 4y_C &\leq y
\end{aligned}$$

and then push this y as small as you can!

LP for P2

- Finally, the LP of P2's optimal strategy reads

$$\begin{aligned} \min_{y_A, y_B, y_C, y} \quad & y \\ \text{s.t.} \quad & y_A + y_B + y_C = 1 \\ & y_A \geq 0 \\ & y_B \geq 0 \\ & y_C \geq 0 \\ & 3y_A + y_B - y_C \leq y \\ & -2y_A + 3y_B + 2y_C \leq y \\ & y_A - 2y_B + 4y_C \leq y \end{aligned}$$

- Solving this LP yields

$$y_A = y_B = y_C = \frac{1}{3}, \quad y = 1$$

which means if P2 choose A, B and C with equal prob. $\frac{1}{3}$, then regardless of what P1 choose, P1 will gain no higher than $y = 1$ in expectation.

They are dual!

- LP for P1 and P2

$$\begin{aligned}\max_{x_A,x_B,x_C,x} \quad & x \\ \text{s.t.} \quad & x_A + x_B + x_C = 1 \\ & x_A \geq 0, x_B \geq 0, x_C \geq 0 \\ & 3x_A - 2x_B + x_C \geq x \\ & x_A + 3x_B - 2x_C \geq x \\ & -x_A + 2x_B + 4x_C \geq x\end{aligned} \qquad \begin{aligned}\min_{y_A,y_B,y_C,y} \quad & y \\ \text{s.t.} \quad & y_A + y_B + y_C = 1 \\ & y_A \geq 0, y_B \geq 0, y_C \geq 0 \\ & 3y_A + y_B - y_C \leq y \\ & -2y_A + 3y_B + 2y_C \leq y \\ & y_A - 2y_B + 4y_C \leq y\end{aligned}$$

  are dual.

- $x \leq y$ (weak duality in general)
- Here we have strong duality $x = y = 1$.
- von Neumann's theorem: it is possible to find an Nash equilibrium in every such game from which it is not worth deviating unilaterally for either of the players, because neither of them can increase their gain this way. The equilibrium for these games can always be achieved by mixed strategies (probabilistic distribution).

  Or in short: there is always a saddle point.

$$\begin{aligned} \max_{x_A, x_B, x_C, x} \quad & x \\ \text{s.t.} \quad & x_A + x_B + x_C = 1 \\ & x_A \geq 0, x_B \geq 0, x_C \geq 0 \\ & 3x_A - 2x_B + x_C \geq x \\ & x_A + 3x_B - 2x_C \geq x \\ & -x_A + 2x_B + 4x_C \geq x \end{aligned}$$

$$\begin{aligned} \min_{y_A, y_B, y_C, y} \quad & y \\ \text{s.t.} \quad & y_A + y_B + y_C = 1 \\ & y_A \geq 0, y_B \geq 0, y_C \geq 0 \\ & 3y_A + y_B - y_C \leq y \\ & -2y_A + 3y_B + 2y_C \leq y \\ & y_A - 2y_B + 4y_C \leq y \end{aligned}$$

$$\begin{aligned} \max_{x_A, x_B, x_C, x} \quad & x \\ \text{s.t.} \quad & \begin{bmatrix} 3 & -2 & 1 \\ 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix} \geq \begin{bmatrix} x \\ x \\ x \end{bmatrix} \\ & x_A \geq 0, x_B \geq 0, x_C \geq 0 \\ & x_A + x_B + x_C = 1 \end{aligned}$$

$$\begin{aligned} \min_{y_A, y_B, y_C, y} \quad & y \\ \text{s.t.} \quad & \begin{bmatrix} 3 & 1 & -1 \\ -2 & 3 & 2 \\ 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} y_A \\ y_B \\ y_C \end{bmatrix} \leq \begin{bmatrix} y \\ y \\ y \end{bmatrix} \\ & y_A \geq 0, y_B \geq 0, y_C \geq 0 \\ & y_A + y_B + y_C = 1 \end{aligned}$$

$$\begin{aligned} \max_{x_A, x_B, x_C, x} \quad & x \\ \text{s.t.} \quad & \begin{bmatrix} -3 & 2 & -1 \\ -1 & -3 & 2 \\ 1 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix} \leq \begin{bmatrix} -x \\ -x \\ -x \end{bmatrix} \\ & x_A \geq 0, x_B \geq 0, x_C \geq 0 \\ & x_A + x_B + x_C = 1 \end{aligned}$$

$$\begin{aligned} \min_{y_A, y_B, y_C, y} \quad & y \\ \text{s.t.} \quad & \begin{bmatrix} -3 & -1 & 1 \\ 2 & -3 & -2 \\ -1 & 2 & -4 \end{bmatrix} \begin{bmatrix} y_A \\ y_B \\ y_C \end{bmatrix} \geq \begin{bmatrix} -y \\ -y \\ -y \end{bmatrix} \\ & y_A \geq 0, y_B \geq 0, y_C \geq 0 \\ & y_A + y_B + y_C = 1 \end{aligned}$$

$$\max_{x_A, x_B, x_C, x} \quad x$$
$$\text{s.t.} \quad \begin{bmatrix} -3 & 2 & -1 & 1 \\ -1 & -3 & 2 & 1 \\ 1 & -2 & -4 & 1 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \\ x_C \\ x \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$
$$x_A \geq 0, x_B \geq 0, x_C \geq 0$$

$$\min_{y_A, y_B, y_C, y} \quad y$$
$$\text{s.t.} \quad \begin{bmatrix} -3 & -1 & 1 & 1 \\ 2 & -3 & -2 & 1 \\ -1 & 2 & -4 & 1 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} y_A \\ y_B \\ y_C \\ y \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$
$$y_A \geq 0, y_B \geq 0, y_C \geq 0$$

$$\max_{x_A, x_B, x_C, x} \quad x^+ - x^-$$
$$\text{s.t.} \quad \begin{bmatrix} -3 & 2 & -1 & 1 & -1 \\ -1 & -3 & 2 & 1 & -1 \\ 1 & -2 & -4 & 1 & -1 \\ 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \\ x_C \\ x^+ \\ x^- \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$
$$x_A \geq 0, x_B \geq 0, x_C \geq 0, x^+ \geq 0, x^- \geq 0$$

$$\min_{y_A, y_B, y_C, y} \quad y^+ - y^-$$
$$\text{s.t.} \quad \begin{bmatrix} -3 & -1 & 1 & 1 & -1 \\ 2 & -3 & -2 & 1 & -1 \\ -1 & 2 & -4 & 1 & -1 \\ 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_A \\ y_B \\ y_C \\ y^+ \\ y^- \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$
$$y_A \geq 0, y_B \geq 0, y_C \geq 0, y^+ \geq 0, y^- \geq 0$$

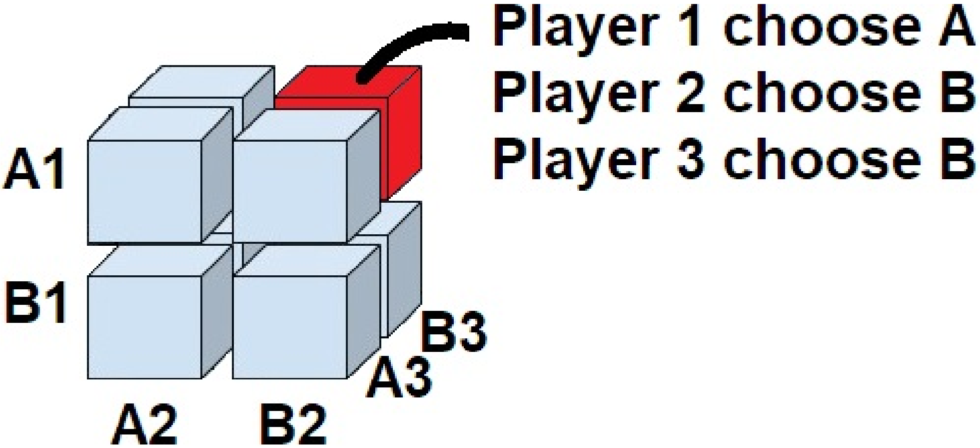
They are dual!

$$\max_{x_A, x_B, x_C, x} \quad x^+ - x^-$$
$$\text{s.t.} \quad \begin{bmatrix} -3 & 2 & -1 & 1 & -1 \\ -1 & -3 & 2 & 1 & -1 \\ 1 & -2 & -4 & 1 & -1 \\ 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \\ x_C \\ x^+ \\ x^- \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$
$$x_A \geq 0, x_B \geq 0, x_C \geq 0, x^+ \geq 0, x^- \geq 0$$

$$\min_{y_A, y_B, y_C, y} \quad y^+ - y^-$$
$$\text{s.t.} \quad \begin{bmatrix} -3 & -1 & 1 & 1 & -1 \\ 2 & -3 & -2 & 1 & -1 \\ -1 & 2 & -4 & 1 & -1 \\ 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_A \\ y_B \\ y_C \\ y^+ \\ y^- \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$
$$y_A \geq 0, y_B \geq 0, y_C \geq 0, y^+ \geq 0, y^- \geq 0$$

- $\mathbf{c} = \mathbf{b} = [0, 0, 0, 1, -1]^\top$, $\mathbf{x}^* = \begin{bmatrix} 0.4737 \\ 0.3158 \\ 0.2105 \\ 177.1268 \\ 176.1268 \end{bmatrix}$, $\mathbf{y}^* = \begin{bmatrix} 0.3333 \\ 0.3333 \\ 0.3333 \\ 186.0092 \\ 185.0092 \end{bmatrix}$, $\mathbf{c}^\top \mathbf{x}^* = 1 = \mathbf{b}^\top \mathbf{y}^*$
- Complementary slackness:
  All $\langle \mathbf{a}^i, \mathbf{x}^* \rangle \leq b_i$ (numerically) tight $\iff$ $y_i \geq 0$ all slack.
  All $\langle \mathbf{a}_i, \mathbf{y}^* \rangle \leq b_i$ (numerically) tight $\iff$ $x_i \geq 0$ all slack.

3-person game



A 2-by-2-by-2 payoff tensor.

Summary

- Super-brief thunder-fast introduction to game theory
- 2-person zero-sum game
- Solving 2-person zero-sum game using LP
- The two player's LP are dual to each other.
- Want to learn more? Free online resource:
  `https://nordstromjf.github.io/IntroGameTheory/frontmatter-1.html`

End of document