CO327 Deterministic OR Models
Boolean logic

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Boolean logic

- Boolean logic variable has only two options: "true" or "false".
- Using binary integer variable, we can encode "true" or "false" as 1 and 0.
- 0-1 BIP can thus be used extensively in logic problems!

"Merchant of Venice" by Shakespeare 1597

- Portia (protagonist): I ♡ smart boys. I put my portrait in one of the following box. I will marry the smart boy who identify which box has the portrait (without opening the box).

- Texts on box:
  - Box-gold: I have the portrait.
  - Box-silver: I do not have the portrait.
  - Box-copper: The portrait is not in box-gold.

- Portia: at most one of the three texts was true.

- Formulate this problem an BIP and solve it.

Decision variable

- For $i = 1, 2, 3$, let

$$x_i = \begin{cases} 1 & \text{if box } i \text{ has the portrait} \\ 0 & \text{else} \end{cases}$$

- Let

$$y_i = \begin{cases} 1 & \text{if the text on box } i \text{ is telling the truth} \\ 0 & \text{else} \end{cases}$$

- Important: you need to recognize that you have TWO sets of decision variables: "is-portrait-in-box" variable and "is-text-true variable".

- If you miss any of the variable, it will make the BIP modeling not work at all $\rightarrow$ that's why modeling is the most important part of applied mathematics (followed by how to solve it).

Modeling the constraint

- Only one box has the portrait

$$x_1 + x_2 + x_3 = 1.$$

- At most one of the three texts was true.

$$y_1 + y_2 + y_3 \leq 1.$$

- Box-1: "I have the portrait"
  - If this is true, $y_1 = 1$ and $x_1 = 1$.
  - If this is false, $y_1 = 0$ and $x_1 = 0$.

  Then we have

$$x_1 = y_1,$$

  or in LP standard form

$$-x_1 + y_1 = 0.$$

Modeling the constraint

- Box-2: "I don't have the portrait"
  - If this is true, $y_2 = 1$ and $x_2 = 0$.
  - If this is false, $y_2 = 0$ and $x_2 = 1$.

  Then we have
  $$x_2 = 1 - y_2,$$
  or in LP standard form
  $$x_2 + y_2 = 1.$$

- Box-3: "The portrait is not in box-1"
  - If this is true, $y_3 = 1$ and $x_1 = 0$.
  - If this is false, $y_3 = 0$ and $x_1 = 1$.

  Then we have
  $$x_1 = 1 - y_3,$$
  or in LP standard form
  $$x_1 + y_3 = 1.$$

Constraint feasibility problem

- There is no preference between feasible solution(s), all feasible solutions are acceptable. $\rightarrow$ no objective function.

- The IP is thus

$$
\begin{array}{ll}
\min & 0 \\
\text{s.t.} & x_1 + x_2 + x_3 = 1 \\
& y_1 + y_2 + y_3 \leq 1 \\
& -x_1 + y_1 = 0 \\
& x_2 + y_2 = 1 \\
& x_1 + y_3 = 1 \\
& x_i, y_i \in \{0, 1\}.
\end{array}
$$

- So, what's the solution to Portia's problem?
  $\rightarrow$ solve this BIP yourself!

“Merchant of Venice”, aftermath

- Later, Portia: one more.
  - Box-gold: The portrait is not in box-silver.
  - Box-silver: I do not have the portrait.
  - Box-copper: I have the portrait.
- At least one of the three texts is true, and at least one of them is false.
- Formulate this problem an BIP and solve it (assignment).

4-person murder (assignment)

- 4 persons, one of whom has committed murder, made the following statements when questioned by the police.
  - Person-A: “B did it”.
  - Person-B: “D did it”.
  - Person-C: “I didn’t do it”.
  - Person-D: “B lied”.

- Who is the guilty person if only 1 of the 4 statements is true? Formulate this as a BIP and solve it.

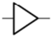
- Who is the guilty person if only 1 of the 4 statements is false? Formulate this as a BIP and solve it.

“One Thousand and One Nights” / “Arabia’s night”

- A king met 3 boys and told them about his mule.
- “What color is the mule?” asked one boy.
- “Either brown, black, or gray. Make a guess.” answer the king.
- Boy-1: “black”.
- Boy-2: “either brown or gray”.
- Boy-3: “brown”.
- The king: “at least one of you guessed right, and at least one of you guessed wrong”.
- Formulate this problem as BIP and find the mule’s color (assignment).

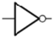
Logic operation / digital computer logic gates

YES



INPUT	OUTPUT
A	
0	0
1	1

NOT



INPUT	OUTPUT
A	
0	1
1	0

AND

INPUT		OUTPUT
A	B	
0	0	0
1	0	0
0	1	0
1	1	1

OR

INPUT		OUTPUT
A	B	
0	0	0
1	0	1
0	1	1
1	1	1

XOR

INPUT		OUTPUT
A	B	
0	0	0
1	0	1
0	1	1
1	1	0

NAND

INPUT		OUTPUT
A	B	
0	0	1
1	0	1
0	1	1
1	1	0

NOR

INPUT		OUTPUT
A	B	
0	0	1
1	0	0
0	1	0
1	1	0

XNOR

INPUT		OUTPUT
A	B	
0	0	1
1	0	0
0	1	0
1	1	1

Logic YES, NOT, AND, OR

- YES
  - Boolean algebra $y = x$
  - IP: $y = x$, $x \in \{0, 1\}$.
- NOT
  - Boolean algebra $y = \neg x$
  - IP: $y = 1 - x$, $x \in \{0, 1\}$.
- AND
  - Boolean algebra $y = x_1 \wedge x_2$
  - IP: ?????
- OR
  - Boolean algebra $y = x_1 \vee x_2$
  - IP: ??????

Logic AND, OR

- AND
  - Boolean algebra $y = x_1 \wedge x_2$
  - IP
    $$y \geq x_1 + x_2 - 1, \quad y \leq x_1, \quad y \leq x_2, \quad y \in \{0, 1\}.$$
    or equivalently
    $$0 \leq x_1 + x_2 - 2y \leq 1.$$
- OR
  - Boolean algebra $y = x_1 \vee x_2$
  - IP
    $$y \leq x_1 + x_2 - 1, \quad y \geq x_1, \quad y \geq x_2, \quad y \in \{0, 1\}.$$
    or equivalently
    $$0 \leq 2y - x_1 - x_2 \leq 1.$$

... dafaq!?

More dafaq

- Logical implication
  - Boolean algebra $y = (x_1 \implies x_2) = \neg x_1 \vee x_2$
  - IP
$$y \le 1 - x_1 + x_2, \quad y \ge 1 - x_1, \quad y \ge x_2, \quad y \in \{0, 1\}.$$
- Forced logical implication
  - Boolean algebra: forcing $x_1 \implies x_2$ must hold
  - IP
$$x_1 \le x_2$$
- XOR
  - Boolean algebra: $y = x_1 \oplus x_2$
  - IP
$$y \le x_1 + x_2, \quad y \ge x_1 - x_2, \quad y \ge x_2 - x_1 \quad y \le 2 - x_1 - x_2, \quad y \in \{0, 1\}.$$

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