CO327 Deterministic OR Models
Applications in manufacture industry
Production scheduling and optimal control

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Problem description

- You have a 24-hour working, fully automated factory that continuously making a type product (let it be $\mathbb{P}$).
- You run the factory for T weeks (after that you take a pause for repairing the gears and take a break).
- You receive orders from buyers during this T weeks. Due to market fluctuations, the amount of demands are changing everyday.
- You signed contract with these buyers that the factory has to produce $\mathbb{P}$ to meet all the demands, no backlog (you go to jail if you can't make it supply enough $\mathbb{P}$ on time).

Production rate and demand rate

- You are provided with sufficient (infinite) amount of raw material and resources (power, electricity) to operate the factory.
- What you can do is to adjust how fast the machines operate on producing $\mathbb{P}$. For example, you can make the machine run faster in "turbo mode", or you can slow down the machine.
- Machine working in turbo mode will get some damages, buy taking these damages as cost, you get a different "product cost" for the machine working in different rate.
- Control problem: determine the production rate such that all the demands are stratified during this T weeks.

More problem description: physical limitations

- The machine working speed has a limit: from 0 (closed) to b_1 (maximum speed).
- If your factory produces surplus $\mathbb{P}$ (over the demand), you can store them temporarily in the storage space. Your storage capacity is b_2.

More problem description: rates and inventory

- Let the demand rate (rate of change of demand) at time t be $g(t)$.
- Let the production rate (range of change of production) at time t be $p(t)$.
- Let the inventory (amount of product $\mathbb{P}$) at time t be $P(t)$.
- Boundary conditions:
  - Initially you have P_0 amount of $\mathbb{P}$ in storage.
  - At time T, the desired amount of $\mathbb{P}$ in storage is P_T.
- The relationship between p, g, P at time t is

$$P(t) = P_0 + \int_0^t \Big(p(\tau) - g(\tau)\Big) d\tau, \quad t \in [0, T]$$

Problem description: storage cost

- Storing the surplus $\mathbb{P}$ costs money.
- For example, if $\mathbb{P}$ is frozen food, then storing these $\rightarrow$ electricity bills.
- Therefore, a part of operational cost comes from $P(t)$: storage cost.
- Assume that the inventory storage cost is proportional to the unit storage, then the total cost for this T weeks is

$$\text{inventory storage cost} = c_1 \int_0^T P(t)dt,$$

where $c_1 > 0$ is known.

Problem description: production operational cost

- Production itself has a cost.
- For example, the machine costs electricity to run.
- Although you are provided with infinite ϟ, but ϟ is not free (of course!)
- Assumes the production operational cost $\propto$ the production rate, the total cost in this T weeks is then

$$\text{production operational cost} = c_2 \int_0^T p(t)dt,$$

  where $c_2 > 0$ is known.
- i.e., for example, if you run the machine in turbo mode, it costs more ϟ $\rightarrow$ more cost!

Problem description: total cost

- Now the total cost you want to minimize is

$$
\begin{aligned}
c(t) &= \text{inventroy storage cost} + \text{production operational cost} \\
&= c_1 \int_0^T P(t)dt + c_2 \int_0^T p(t)dt \\
&= \int_0^T \Big(c_1 P(t) + c_2 p(t) \Big) dt.
\end{aligned}
$$

- So the optimization is

$$
\begin{aligned}
\min_{p(t)} \quad & \int_0^T \Big(c_1 P(t) + c_2 p(t) \Big) dt \\
\text{s.t.} \quad & \text{constraints in } p(t) \text{ and } P(t)
\end{aligned}
$$

The minimization problem with constraints

- The complete model is

$$
\begin{aligned}
\min_{p(t)} \quad & \int_0^T \Big(c_1 P(t) + c_2 p(t)\Big) dt \\
\text{s.t.} \quad & P(t) = P_0 + \int_0^t \Big(p(\tau) - g(\tau)\Big) d\tau \quad t \in [0, T] \\
& P(T) = P_T \\
& 0 \le p(t) \le b_1 \qquad t \in [0, T] \\
& 0 \le P(t) \le b_2 \qquad t \in [0, T]
\end{aligned}
$$

- Constraints are
  - The relationship between p, g, P
  - Boundary condition of inventory at T: P_T
  - Machine production rate in $[0, b_1]$
  - Inventory storage in $[0, b_2]$

Optimal control

- The problem is an optimal control problem

$$
\begin{aligned}
\min_{p(t)} \quad & \int_0^T \Big(c_1 P(t) + c_2 p(t)\Big) dt \\
\text{s.t.} \quad & P(t) = P_0 + \int_0^t \Big(p(\tau) - g(\tau)\Big) d\tau \quad & t \in [0, T] \\
& P(T) = P_T \\
& 0 \leq p(t) \leq b_1 & t \in [0, T] \\
& 0 \leq P(t) \leq b_2 & t \in [0, T]
\end{aligned}
$$

- You can control $p(t)$ directly, which is called *control variable*.
- You cannot control $P(t)$ directly, but it is *influenced* by the control variable, so it is called a state variable.
- Optimal control theory $=$a branch of mathematical optimization that deals with finding a control for a dynamical system over a period of time such that an objective function is optimized.

Linear optimal control

- The mathematics prerequisites of general optimal control are ODE, Dynamical system and Calculus of Variations.

- None of them will appear in this course $\rightarrow$ we do approximation!
  - Discretization: discretize the time $[0, T]$ to $[0, \Delta, 2\Delta, ..., n\Delta]$.
  - Linearization: linearize $P(t)$ and $p(t)$, assuming they are constant linear function over each Δ fragment

$$
\begin{aligned}
\min_{p_j} \quad & \sum_{j=1}^{n} \Big(c_1 P_j + c_2 p_j\Big)\Delta \\
\text{s.t.} \quad & P_j = P_{j-1} + \Big(p_j - g_j\Big)\Delta \quad && j \in [1, n] \\
& P_n = P_T \\
& 0 \leq p_j \leq b_1 && j \in [1, n] \\
& 0 \leq P_j \leq b_2 && j \in [1, n]
\end{aligned}
$$

- What is optimal control: determine the control sequence $p_1, p_2, \cdots, p_n$ to optimize your system (total operational cost here).
  $\Longrightarrow$ Here decision variable is a time series.

Other issue

- In fact, the total cost

$$c = \sum_{j=1}^{n} \Big(c_1 P_j + c_2 p_j \Big) \Delta$$

  is incomplete, for example, it missed the *rental cost*: your factory site is not a free asset.

- So in reality, you need to include these costs.

- However, we do not need to include such rental cost in the formulation of optimal control, because
  - the rent is probably fix (= constant)
  - even if rent is not a constant, this value is independent of production rate p (should be right?).

Summary

- Linear programming and its application to (linear discretized) optimal control.
- An example that the decision variable is a time series.

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