

CO327 Deterministic OR Models

Optimal transport and resource allocation

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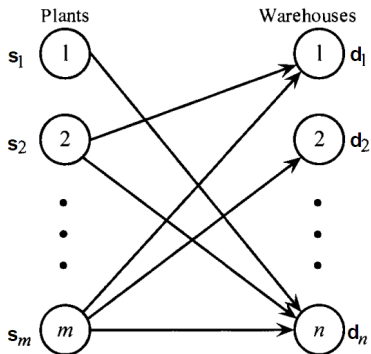
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Monge-Kantorovich-Hitchcock-Koopmans transport problem

- ▶ You have a coffee company that processes coffee beans into coffee at m plants.
- ▶ The coffee is then shipped every week to n warehouses in major cities for retail, distribution, and exporting.
- ▶ Suppose the unit shipping cost from plant i to warehouse j is c_{ij} . Here the unit transportation cost is independent of the number of goods transported.
- ▶ Suppose that the production capacity at plant i is s_i and that the demand at warehouse j is d_j .
- ▶ Find the production-shipping pattern x_{ij} from plant i to warehouse j , $i = 1, \dots, m, j = 1, \dots, n$, which minimizes the overall shipping cost.

Matching a bipartite graph



- Bipartite graph (bigraph): a graph whose vertices can be divided into two disjoint and independent sets (“Plants” and “Warehouses” here) such that every edge connects a vertex in a set to another one.

Transportation matrix

		Destination j				
		1	2	\dots	n	
source i	1	c_{11}	c_{12}	\dots	c_{1n}	s_1
	2	c_{21}	c_{22}	\dots	c_{2n}	s_2
	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
	m	c_{m1}	c_{m2}	\dots	c_{mn}	s_m
		d_1	d_2	\dots	d_n	

- ▶ There are m sources.
- ▶ Each source has a supply upper limit s_i .
- ▶ There are n destinations.
- ▶ Each destination has a demand d_j .

The decision variable

		Destination j				
		1	2	\dots	n	
source i	1	x_{11}	x_{12}	\dots	x_{1n}	s_1
	2	x_{21}	x_{22}	\dots	x_{2n}	s_2
	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
	m	x_{m1}	x_{m2}	\dots	x_{mn}	s_m
		d_1	d_2	\dots	d_n	

- ▶ x_{ij} = production-shipping pattern from source i to destination j
- ▶ Nonnegativity $x_{ij} \geq 0$.
- ▶ If integral amount $x_{ij} \in \mathbb{N}$.
- ▶ The objective function: the total transportation cost

$$\sum_{ij} c_{ij} x_{ij} = \text{Tr } \mathbf{C}^\top \mathbf{X} = \langle \mathbf{C}, \mathbf{X} \rangle.$$

- ▶ Goal: minimize.

The constraints

		Destination j				
		1	2	\dots	n	
source i	1	x_{11}	x_{12}	\dots	x_{1n}	s_1
	2	x_{21}	x_{22}	\dots	x_{2n}	s_2
	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
	m	x_{m1}	x_{m2}	\dots	x_{mn}	s_m
		d_1	d_2	\dots	d_n	

- ▶ Meets the demand

$$\sum_i x_{ij} = d_j, j = 1, \dots, n$$

i.e., column sum of \mathbf{X} meets d_j : $\mathbf{X}^\top \mathbf{1} = \mathbf{d}$.

- ▶ Meets the supply

$$\sum_j x_{ij} = s_i, i = 1, \dots, m$$

i.e., row sum of \mathbf{X} meets s_i : $\mathbf{X}\mathbf{1} = \mathbf{s}$.

LP of optimal transport

$$\begin{aligned} \min_{x_{ij}} \quad & \sum_{ij} c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_j x_{ij} = s_j, i = 1, \dots, m \\ & \sum_i x_{ij} = d_j, j = 1, \dots, n \\ & x_{ij} \geq 0, \text{ for all } i, j \end{aligned}$$
$$\begin{aligned} \min_{\mathbf{X}} \quad & \langle \mathbf{C}, \mathbf{X} \rangle \\ \text{s.t.} \quad & \mathbf{X}\mathbf{1} = \mathbf{s} \\ & \mathbf{X}^\top \mathbf{1} = \mathbf{d} \\ & \mathbf{X} \geq 0 \end{aligned}, \mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{bmatrix}, \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

- ▶ All values: constants s, d, c and variable x are nonnegative.
- ▶ Optimal transport is to find the optimal x_{ij} pattern.

More modeling: balanced vs unbalanced

- ▶ Balanced transport problem

$$\sum_i \text{supply}_i = \sum_j \text{demand}_j.$$

- ▶ Unbalanced transport problem

$$\sum_i \text{supply}_i \neq \sum_j \text{demand}_j.$$

- ▶ Case 1. $\sum_i \text{supply}_i > \sum_j \text{demand}_j$
- ▶ Case 2. $\sum_i \text{supply}_i < \sum_j \text{demand}_j$

Unbalanced transport problem: more supply than demand

- ▶ More supply than demands \implies some sources can “slow-down”
- ▶ Suppose there is no penalty in slowing down the machine in sources, then the LP becomes

$$\begin{aligned} \min_{x_{ij}} \quad & \sum_{ij} c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_j x_{ij} \leq s_j, i = 1, \dots, m \\ & \sum_i x_{ij} = d_j, j = 1, \dots, n \\ & x_{ij} \geq 0, \text{ for all } i, j \end{aligned}$$

What it means: it is not necessary for all sources to work at FULL production-capacity.

- ▶ Note: here demands are still satisfied (the demand constraints are equality).

Unbalanced transport problem: more demand than supply

- ▶ More demand than supply \implies some destinations do not get FULL shipment
- ▶ Suppose there is no penalty for incomplete shipment, then the LP becomes

$$\begin{aligned} \min_{x_{ij}} \quad & \sum_{ij} c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_j x_{ij} = s_j, i = 1, \dots, m \\ & \sum_i x_{ij} \leq d_j, j = 1, \dots, n \\ & x_{ij} \geq 0, \text{ for all } i, j \end{aligned}$$

- ▶ Note: here supplies are still running at FULL production-capacity (the supply constraints are equality).

Case supply $>$ demand, with penalty

- ▶ Suppose now sources are forced to produce at FULL capacity, and there is a penalty of surplus products.
- ▶ How to model: add a new demand “black hole” that sucks all the surplus products to the void

		Destination j				Black hole	
		1	2	\dots	n		
source i	1	c_{11}	c_{12}	\dots	c_{1n}	$c_{1,n+1}$	s_1
	2	c_{21}	c_{22}	\dots	c_{2n}	$c_{2,n+1}$	s_2
	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots
	m	c_{m1}	c_{m2}	\dots	c_{mn}	$c_{m,n+1}$	s_m
		d_1	d_2	\dots	d_n	d_{n+1}	

with $c_{i,n+1}$ as the cost of sending the surplus products to this black hole (which is a waste).

- ▶ The demand of black hole is then exactly $d_{n+1} = \sum_{i=1}^m s_i - \sum_{j=1}^n d_j$, no more, no less.

Case demand $>$ supply, with penalty

- ▶ Suppose now backlog are not allowed: we have to meet all the demands, and there is a penalty of deficient supply.
- ▶ How to model: add a new supply “white hole” that produces all the deficient products from the void

		Destination j				
		1	2	\dots	n	
source i	1	c_{11}	c_{12}	\dots	c_{1n}	s_1
	2	c_{21}	c_{22}	\dots	c_{2n}	s_2
	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
	m	c_{m1}	c_{m2}	\dots	c_{mn}	s_m
White hole		$c_{m+1,1}$	$c_{m+1,2}$	\dots	$c_{m+1,n}$	s_{m+1}
		d_1	d_2	\dots	d_n	

with $c_{m+1,i}$ as the cost of sending the products from this white hole to destination i .

- ▶ The demand of black hole is then exactly $s_{m+1} = \sum_{j=1}^n d_j - \sum_{i=1}^m s_i$, no more, no less.

More modeling: unit transportation cost varies

- ▶ Previously, c_{ij} = the unit transportation cost from source i to destination j , such cost is independent of the number of goods transported.
 - ▶ i.e., transporting 10 goods from source i to destination j costs $10c_{ij}$.
 - ▶ i.e., transporting 99999 goods from source i to destination j costs $99999c_{ij}$.
- ▶ It is possible that the unit transportation cost varies with the number of goods transported.
- ▶ For example if c_{ij} scale linearly with x_{ij} , e.g.

$$c_{ij} = p_{ij}x_{ij}$$

where p_{ij} is a constant, then the transportation cost from source i to j becomes $p_{ij}x_{ij}^2 \implies$ Quadratic program.

- ▶ In general, c_{ij} can varies with x_{ij} as crazy as you want \implies nonlinear program.

Summary

- ▶ The transport problem
- ▶ Transport problem = matching a bipartite graph
- ▶ The transportation matrix
- ▶ Optimal transport = optimal pattern of x_{ij}
- ▶ Optimal transport = Wasserstein distance
- ▶ Balanced and unbalanced problem
- ▶ The black hole / white hole trick.

History

- ▶ The optimal transport we saw is the classical (old) form of transportation theory:
 - ▶ First formalized by French mathematician Gaspard Monge in 1781.
 - ▶ Major advances during World War II by the Soviet mathematician and economist Leonid Kantorovich.
 - ▶ Hitchcock & Koopman: linear programming formulation.
 - ▶ Kantorovich and Koopman won the Nobel Prize in Economics 1975.
- ▶ Next: to go example.

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Extra slide: Wasserstein distance

- ▶ The Wasserstein distance d between \mathbf{p} and \mathbf{q} with respect to \mathbf{C} is defined as

$$d_W(\mathbf{p}, \mathbf{q}) := \min_{\mathbf{X}} \langle \mathbf{C}, \mathbf{X} \rangle \quad \text{s.t.} \quad \begin{aligned} \mathbf{X}\mathbf{1} &= \mathbf{p} \\ \mathbf{X}^\top \mathbf{1} &= \mathbf{q} \\ \mathbf{X} &\geq 0 \end{aligned}$$

- ▶ Geometry interpretation: $\langle \mathbf{C}, \mathbf{X} \rangle = f$ is a hyperplane and

$$\mathbf{X}\mathbf{1} = \mathbf{p}, \mathbf{X}^\top \mathbf{1} = \mathbf{q}, \mathbf{X} \geq 0$$

is a polyhedron. So Wasserstein distance is just a LP.

- ▶ Resource allocation interpretation: the optimal transport cost of a city if we change the supply-demand specification.
- ▶ Other name of optimal transport: Kantorovich - Rubinstein, Earth mover's distance.