CO327 Deterministic OR Models
Optimal transport and resource allocation

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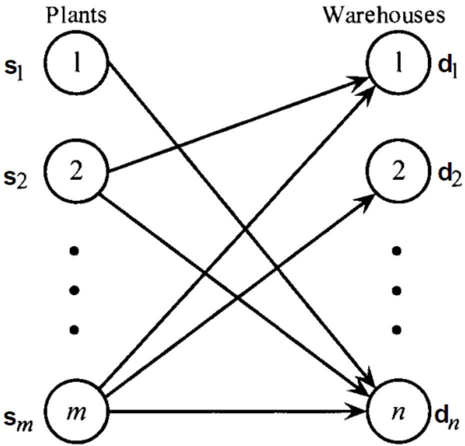
Monge-Kantorovich-Hitchcock-Koopmans transport problem

- You have a coffee company that processes coffee beans into coffee at m plants.
- The coffee is then shipped every week to n warehouses in major cities for retail, distribution, and exporting.
- Suppose the unit shipping cost from plant i to warehouse j is c_{ij}. Here the unit transportation cost is independent of the number of goods transported.
- Suppose that the production capacity at plant i is s_i and that the demand at warehouse j is d_j.
- Find the production-shipping pattern x_{ij} from plant i to warehouse j, $i = 1, ..., m, j = 1, ..., n$, which minimizes the overall shipping cost.

Matching a bipartite graph



- Bipartite graph (bigraph): a graph whose vertices can be divided into two disjoint and independent sets ("Plants" and "Warehouses" here) such that every edge connects a vertex in a set to another one.

Transportation matrix

$$
\begin{array}{cc|c|c|c|c|c}
 & & \multicolumn{4}{c}{\text{Destination } j} & \\
 & & 1 & 2 & \cdots & n & \\
\cline{3-6}
 & 1 & c_{11} & c_{12} & \cdots & c_{1n} & s_1 \\
\cline{3-6}
 & 2 & c_{21} & c_{22} & \cdots & c_{2n} & s_2 \\
\cline{3-6}
\text{source } i & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\cline{3-6}
 & m & c_{m1} & c_{m2} & \cdots & c_{mn} & s_m \\
\cline{3-6}
 & & d_1 & d_2 & \cdots & d_n &
\end{array}
$$

- The are m sources.
- Each source as a supply upper limit s_i.
- There are n destinations.
- Each destination has a demand d_j.

The decision variable

$$
\begin{array}{cc|c|c|c|c|c}
 & \multicolumn{6}{c}{\text{Destination } j} \\
 & & 1 & 2 & \cdots & n & \\
\cline{3-6}
 & 1 & x_{11} & x_{12} & \cdots & x_{1n} & s_1 \\
\cline{3-6}
 & 2 & x_{21} & x_{22} & \cdots & x_{2n} & s_2 \\
\cline{3-6}
\text{source } i & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\cline{3-6}
 & m & x_{m1} & x_{m2} & \cdots & x_{mn} & s_m \\
\cline{3-6}
 & & d_1 & d_2 & \cdots & d_n &
\end{array}
$$

- x_{ij} = production-shipping pattern from source i to destination j
- Nonnegativity $x_{ij} \geq 0$.
- If integral amount $x_{ij} \in \mathbb{N}$.
- The objective function: the total transportation cost

$$\sum_{ij} c_{ij} x_{ij} = \operatorname{Tr} \mathbf{C}^\top \mathbf{X} = \langle \mathbf{C}, \mathbf{X} \rangle.$$

- Goal: minimize.

The constraints

$$
\begin{array}{r c|c|c|c|c|c}
 & & \multicolumn{4}{c}{\text{Destination } j} & \\
 & & 1 & 2 & \cdots & n & \\
\cline{3-6}
 & 1 & x_{11} & x_{12} & \cdots & x_{1n} & s_1 \\
\cline{3-6}
 & 2 & x_{21} & x_{22} & \cdots & x_{2n} & s_2 \\
\cline{3-6}
\text{source } i & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\cline{3-6}
 & m & x_{m1} & x_{m2} & \cdots & x_{mn} & s_m \\
\cline{3-6}
 & & d_1 & d_2 & \cdots & d_n &
\end{array}
$$

- Meets the demand

$$\sum_i x_{ij} = d_j, j = 1, ..., n$$

i.e., column sum of $\mathbf{X}$ meets d_j: $\mathbf{X}^\top \mathbf{1} = \mathbf{d}$.

- Meets the supply

$$\sum_j x_{ij} = s_i, i = 1, ..., m$$

i.e., row sum of $\mathbf{X}$ meets s_i: $\mathbf{X}\mathbf{1} = \mathbf{s}$.

LP of optimal transport

$$
\begin{aligned}
\min_{x_{ij}} \quad & \sum_{ij} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{j} x_{ij} = s_i, i = 1, ..., m \\
& \sum_{i} x_{ij} = d_j, j = 1, ..., n \\
& x_{ij} \geq 0, \text{ for all } i, j
\end{aligned}
\qquad
\begin{aligned}
\min_{\mathbf{X}} \quad & \langle \mathbf{C}, \mathbf{X} \rangle \\
\text{s.t.} \quad & \mathbf{X}\mathbf{1} = \mathbf{s} \\
& \mathbf{X}^\top \mathbf{1} = \mathbf{d} \\
& \mathbf{X} \geq 0
\end{aligned}
, \mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{bmatrix}, \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}
$$

- All values: constants s, d, c and variable x are nonnegative.
- Optimal transport is to find the optimal x_{ij} pattern.

More modeling: balanced vs unbalanced

- Balanced transport problem

$$\sum_i \text{supply}_i = \sum_j \text{demand}_j.$$

- Unbalanced transport problem

$$\sum_i \text{supply}_i \neq \sum_j \text{demand}_j.$$

  - Case 1. $\sum_i \text{supply}_i > \sum_j \text{demand}_j$
  - Case 2. $\sum_i \text{supply}_i < \sum_j \text{demand}_j$

Unbalanced transport problem: more supply than demand

- More supply than demands $\implies$ some sources can "slow-down"
- Suppose there is no penalty in slowing down the machine in sources, then the LP becomes

$$\begin{aligned} \min_{x_{ij}} \quad & \sum_{ij} c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{j} x_{ij} \leq s_i, i = 1, \dots, m \\ & \sum_{i} x_{ij} = d_j, j = 1, \dots, n \\ & x_{ij} \geq 0, \text{ for all } i, j \end{aligned}$$

What it means: it is not necessary for all sources to work at FULL production-capacity.

- Note: here demands are still satisfied (the demand constraints are equality).

Unbalanced transport problem: more demand than supply

- More demand than supply $\Longrightarrow$ some destinations do not get FULL shipment
- Suppose there is no penalty for incomplete shipment, then the LP becomes

$$
\begin{aligned}
\min_{x_{ij}} \quad & \sum_{ij} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{j} x_{ij} = s_i, i = 1, ..., m \\
& \sum_{i} x_{ij} \leq d_j, j = 1, ..., n \\
& x_{ij} \geq 0, \text{ for all } i, j
\end{aligned}
$$

- Note: here supplies are still running at FULL production-capacity (the supply constraints are equality).

Case supply > demand, with penalty

- Suppose now sources are forced to produce at FULL capacity, and there is a penalty of surplus products.
- How to model: add a new demand "black hole" that sucks all the surplus products to the void

Destination j

		1	2	$\cdots$	n	Black hole	
source i	1	c_{11}	c_{12}	$\cdots$	c_{1n}	$c_{1,n+1}$	s_1
	2	c_{21}	c_{22}	$\cdots$	c_{2n}	$c_{2,n+1}$	s_2
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
	m	c_{m1}	c_{m2}	$\cdots$	c_{mn}	$c_{m,n+1}$	s_m
		d_1	d_2	$\cdots$	d_n	d_{n+1}	

with $c_{i,n+1}$ as the cost of sending the surplus products to this black hole (which is a waste).

- The demand of black hole is then exactly $d_{n+1} = \sum_{i=1}^{m} s_i - \sum_{j=1}^{n} d_j$, no more, no less.

Case demand > supply, with penalty

- Suppose now backlog are not allowed: we have to meet all the demands, and there is a penalty of deficient supply.
- How to model: add a new supply "white hole" that produces all the deficient products from the void

$$
\begin{array}{cc|c|c|c|c|c}
 & & \multicolumn{4}{c}{\text{Destination } j} & \\
 & & 1 & 2 & \cdots & n & \\
\cline{3-6}
 & 1 & c_{11} & c_{12} & \cdots & c_{1n} & s_1 \\
\cline{3-6}
\text{source } i & 2 & c_{21} & c_{22} & \cdots & c_{2n} & s_2 \\
\cline{3-6}
 & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\cline{3-6}
 & m & c_{m1} & c_{m2} & \cdots & c_{mn} & s_m \\
\cline{3-6}
 & \text{White hole} & c_{m+1,1} & c_{m+1,2} & \cdots & c_{m+1,n} & s_{m+1} \\
 & & d_1 & d_2 & \cdots & d_n &
\end{array}
$$

with $c_{m+1,i}$ as the cost of sending the products from this white hole to destination i.

- The demand of black hole is then exactly $s_{m+1} = \sum_{j=1}^{n} d_j - \sum_{i=1}^{m} s_i$, no more, no less.

Summary

- The transport problem
- Transport problem = matching a bipartite graph
- The transportation matrix
- Optimal transport = optimal pattern of x_{ij}
- Balanced and unbalanced problem
- The black hole / white hole trick
- Extra
  - History
  - More modelling
  - Optimal transport = Wasserstein distance

Extra slide: History

- The optimal transport we saw is the classical (old) form of transportation theory:
  - First formalized by French mathematician Gaspard Monge in 1781.
  - Major advances during World War II by the Soviet mathematician and economist Leonid Kantorovich.
  - Hitchcock & Koopman: linear programming formulation.
  - Kantorovich and Koopman won the Nobel Prize in Economics 1975.

Extra slide: More modelling – unit transportation cost varies

- Previously, c_{ij}= the unit transportation cost from source i to destination j, such cost is independent of the number of goods transported.
  - i.e., transporting 10 goods from source i to destination j costs $10c_{ij}$.
  - i.e., transporting 99999 goods from source i to destination j costs $99999c_{ij}$.
- It is possible that the unit transportation cost varies with the number of goods transported.
- For example if c_{ij} scale linearly with x_{ij}, e.g.

$$c_{ij} = p_{ij}x_{ij}$$

  where p_{ij} is a constant, then the transportation cost from source i to j becomes $p_{ij}x_{ij}^2 \implies$ Quadratic program.
- In general, c_{ij} can varies with x_{ij} as crazy as you want $\implies$ nonlinear program.

Extra slide: Wasserstein distance

- The Wasserstein distance d between $\mathbf{p}$ and $\mathbf{q}$ with respect to $\mathbf{C}$ is defined as

$$d_W(\mathbf{p},\mathbf{q}) := \min_{\mathbf{X}} \quad \langle \mathbf{C}, \mathbf{X} \rangle \quad \text{s.t.} \quad \begin{aligned} &\mathbf{X}\mathbf{1} = \mathbf{p} \\ &\mathbf{X}^\top \mathbf{1} = \mathbf{q} \\ &\mathbf{X} \geq \mathbf{0} \end{aligned}$$

- Geometry interpretation: $\langle \mathbf{C}, \mathbf{X} \rangle = f$ is a hyperplane and

$$\mathbf{X}\mathbf{1} = \mathbf{p}, \mathbf{X}^\top \mathbf{1} = \mathbf{q}, \mathbf{X} \geq \mathbf{0}$$

  is a polyhedron. So Wasserstein distance is just a LP.

- Resource allocation interpretation: the optimal transport cost of a city if we change the supply-demand specification.

- Other name of optimal transport: Kantorovich - Rubinstein, Earth mover's distance.

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