CO327 Deterministic OR Models
Modern optimal transport in machine learning

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First draft: Janurary 3, 2021
Last update: January 3, 2021

Modern optimal transport in machine learning

- LP of optimal transport

$$\begin{aligned} \min_{\mathbf{X}} \quad & \langle \mathbf{C}, \mathbf{X} \rangle \\ \text{s.t.} \quad & \mathbf{X}\mathbf{1} = \mathbf{s} \\ & \mathbf{X}^\top \mathbf{1} = \mathbf{d} \\ & \mathbf{X} \geq 0 \end{aligned}$$

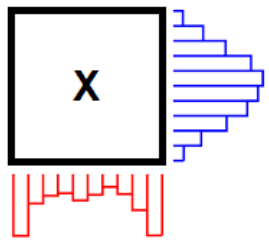
- We previously go through the classical optimal transport in the lens of resource allocation problem.
- Now we look at the modern optimal transport in machine learning.

Visualizing the transport matrix

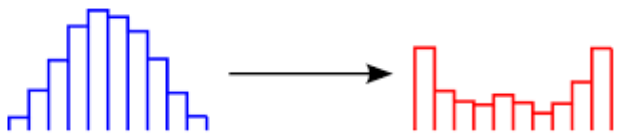
- Recall the transportation matrix

source i \ Destination j	1	2	$\cdots$	n	
1	c_{11}	c_{12}	$\cdots$	c_{1n}	s_1
2	c_{21}	c_{22}	$\cdots$	c_{2n}	s_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
m	c_{m1}	c_{m2}	$\cdots$	c_{mn}	s_m
	d_1	d_2	$\cdots$	d_n	

- Plotting supply and demand as histograms.



- We can see that optimal transport is to create a "mapping" between two histograms

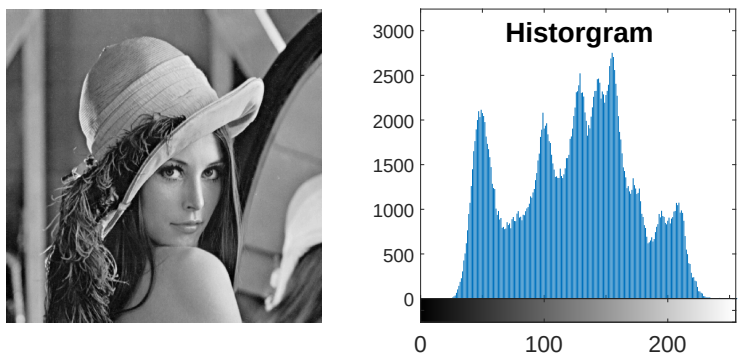


Application: grayscale image histogram adjustment

- A grayscale image $\mathbf{X}$ is a matrix in $\{1, \ldots, 256\}^{m \times n}$, where $m=$ number of rows and $n=$number of columns.

- The x_{ij} is the pixel at the ith row and jth column, and its value is the grayscale value (1-256), with $1=$black, $256=$white and the values in between are gray.

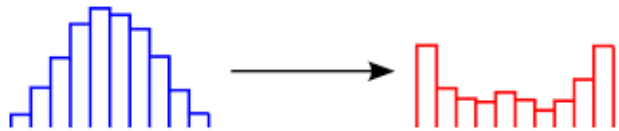
- $\mathbf{X}$ has mn values between 1 and 256. We can count how many of each value appears in the image and form a histogram.



The grayscale "Lenna" image and its historgram.

Grayscale image histogram adjustment = optimal transport

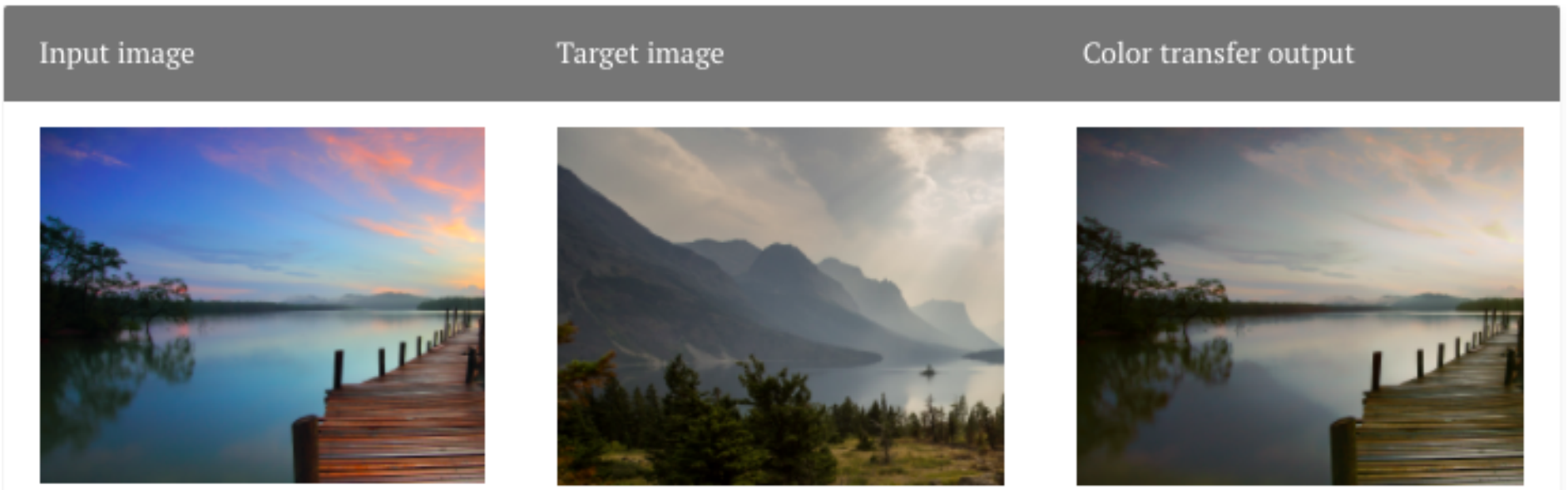
- By performing OT on the histogram, we can modify the "tone", "luminescence" of the image!
- The input image provided can be treated as source in the OT model.



- So what we need are
  - a destination histogram, in which
    - we can made it up
    - we can use the histogram of another image
  - The cost matrix $\mathbf{C}$, in which we need to define.
    - we can just take it as simple as $c_i = x_i - y_i$ where x_i comes from the source histogram and y_i comes from the destination histogram.
    - in general we can take it as $c_i = f(x_i - y_i)$, where f can be any complicated function you want
- The effect of the image adjustment depends on the choice of f and the choice of destination histogram.

Illustrations

Lenna image after OT on histogram equalization. Source: Numerical-tours.



On RGB color image. Source: dcoeurjo.github.io/OTColorTransfer.