CO327 Deterministic OR Models (2021-Spring)

Set covering: location decision and urban planning

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Fire station location problem

You build fire station to put out fire. Image modeified from the internet.

Fire station location problem

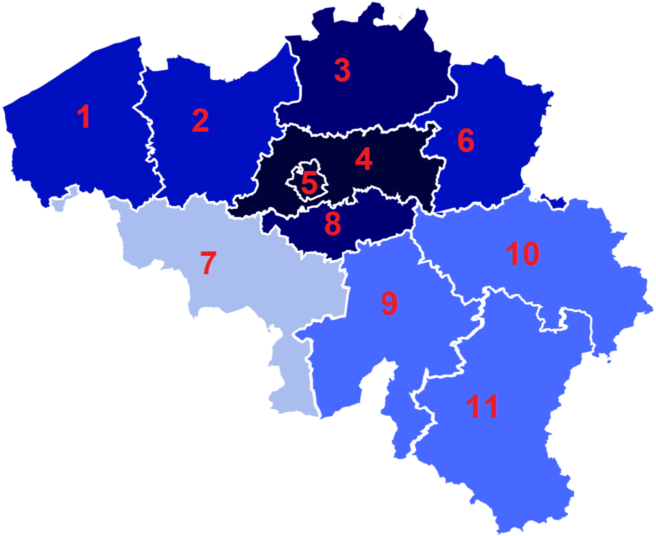
- The best sol. to prevent fire: place fire stations everywhere!
  Too expensive, impossible.
- Fire station location problem: decide where to place the fire station, such that the total cost is minimized, while all city region are under "enough" protection.
- A binary integer programming problem.
  A Set cover problem!

Belgium

Antwerp
West Flanders
East Flanders
Limburg
Flemish
Brabant
Brussels
Walloon
Brabant
Hainaut
Liège
Namur
Luxembourg

Fire station location problem

- Suppose we consider the fire station location problem in Belgium.
- First, label the provinces



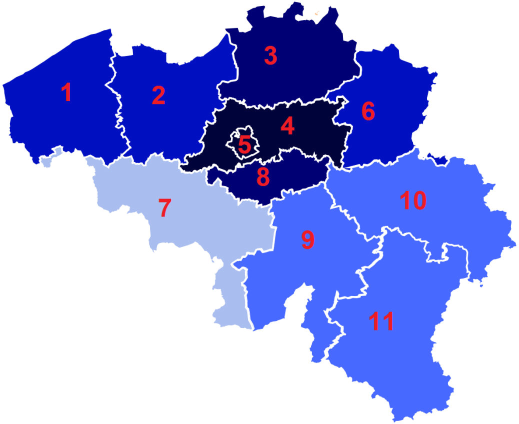
Decision variable and cost function

- Let $x_1, \dots, x_{11} \in \{0, 1\}$ be the decision variable.
- $x_1 = 1$ means place a fire station in 1.
- $x_7 = 0$ means do not place a fire station in 7.
- Cost = total number of fire station built

$$x_1 + x_2 + \cdots + x_{11}.$$

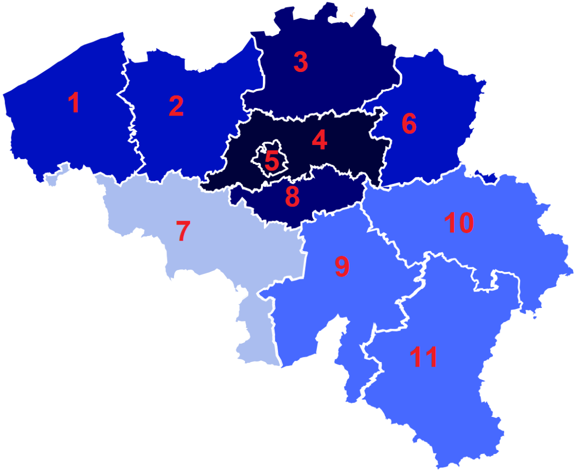
- Goal: minimize this number subject to constraints.

Constraint specification



- Each province needs to be covered by 2 fire station.
- All fire stations are capable to put out all the fire of its own province, plus all its surrounding contacting provinces (assume the fire fighters are supermen).

Adjacency matrix



$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Constraint: $\mathbf{A}\mathbf{x} \geq \mathbf{2}$.

LP of fire station location problem

- In compact notation

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{1}^\top \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{2} \\ & \mathbf{x} \in \{0,1\}^{11} \end{aligned}$$

- We saw that the key in location problem is the formulation of the adjacency matrix $\mathbf{A}$ and the constraint $\geq \mathbf{2}$.

- Recall the Tower Defense problem: it is the same that, the hardest part in modeling the TD problem is to identify the adjacency matrix.

More complicated way to construct the adjacency matrix

- Suppose there are only 4 provinces $x_1, \cdots, x_4$. And for each province you need to ensure that at least 2 fire stations will be located within 15 minutes of travel. The time (minute) of traveling between the provinces, which can be non-symmetrical:

	To			
From	1	2	3	4
1	0	5	20	25
2	5	0	15	18
3	20	15	0	13
4	20	12	13	0

- The adjacency matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

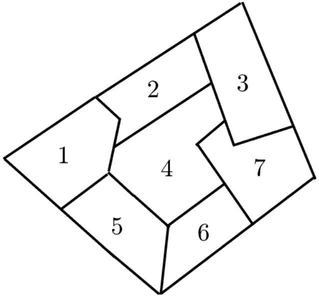
- Note that $\mathbf{A}$ need not to be symmetric.

More complicated scenario: urban planning

The video game “Cities: Skylines”. Source: internet.

Problem description

- You own a city and you can decide what to build there.
- Your task is to decide "what", "where" and "which" to build to maximize people's happiness subject to urban planning constraints.
- The city is divided into 7 regions:



	Land
1	5
2	5
3	5
4	10
5	5
6	4
7	7

where "land" specifies the amount of available space in the region.

Type of buildings

- You are allowed to build 6 type of buildings. Their attributes are listed below

Building	Land needed	Cost	Happiness	Power usage
House	0.2	1.5	0.5	-0.3
Hospital	6	90	1.5	-1.4
Supermarket	1	30	1	-2.5
Shopping mall	10	200	5	-6.6
Cinema	4	10	3.5	-1.2
Power plant	4	10	-10	+10

- In reality: not just 6 types of building.

Looks like a mission impossible

- You have a budget limit of 500.
- Physical constraint: you cannot place buildings over the space limit.
- Physical constraint: you cannot build negative building (anti-matter is not allowed).
- Power constraint: build at least 1 power plant to supply power. All buildings need to have sufficient power supply.
- Power station is equipped with the newest wireless power transfer technology: the power plant can send power to all regions.
- Humanitarian constraint: region 1 to 6 need to have at least 20 houses (you can't let people sleep in the street right?).
- Marketing constraint: at most 1 cinema can be built in each region (too many cinema makes people watching too many movies and not go to work).
  And cinemas cannot be adjacent to each other.
- Humanitarian constraint: for fast emergency service, any hospital has to built in adjacent to all the houses.

Decision variable and objective function

- Decision variable: x_{ij} amount of building i located at region j.
- $x_{ij} \in \mathbb{N}$ and $x_{ij} \geq 0$.
- Simplify the table

Building i	Land	\$	♡	↯
1	0.2	1.5	0.5	-0.3
2	6	90	1.5	-1.4
3	1	30	1	-2.5
4	10	200	5	-6.6
5	4	10	3.5	-1.2
6	4	10	-10	+10

- Happiness from all building 1 in region 1 is $0.5x_{11}$.
  Happiness from all types of building in region 1

$$0.5x_{11} + 1.5x_{21} + x_{31} + 5x_{41} + 3.5x_{51} - 10x_{61}$$

- Objective function: sum of happiness.

Modeling the constraints

- Budget limit of 500

$$\sum_{ij} x_{ij} \$_j \leq 500.$$

- Space limit:

$$\sum_{j} \text{Land}_j x_{ij} \leq \text{Land limit}_j, \quad i = 1, ..., 7$$

- At least 1 power plant

$$\sum_{i} x_{i6} \geq 1.$$

- Power constraint

$$\sum_{ij} x_{ij} \lightning_j \geq 0.$$

- Region 1 to 6 at least 20 hourse

$$\sum_{i=1}^{6} x_{i1} \geq 20.$$

Modeling the constraints

- At most 1 cinema can be build in each region

$$x_{i5} \in \{0, 1\}, i = 1, ..., 7$$

- Cinema cannot be adjacent to each other. According to map, the adjacency matrix is

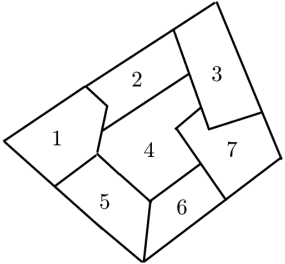
$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

  Constraint is then

$$\mathbf{A}\mathbf{x}_5 \leq 1.$$

Modeling the constraints

- For efficient emergency service, Hospital has to built in a way that it is adjacent to all the houses.



	Land
1	5
2	5
3	5
4	10
5	5
6	4
7	7

- Seems tempting to put $x_{24} = 1$ and that's it. However, if houses only appear in region $1, 4, 6$, then you can also build a hospital in region 5.

- How to model this flexibility? Should we re-determine the decision variable? Should we add new variables? $\rightarrow$ in assignment.

More constraints

- Pollution and garbage distribution.
- Water plant, water storage and distribution.
- Gas plant, gas storage and gas distribution.
- Power transformer and power storage
- Building roads
- Add costs for generating surplus electricity

... who said playing video games is always bad?

Summary

- Fire location problem as set covering problem
- Determining the adjacency matrix is important step in solving set covering problem

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