CO327 Deterministic OR Models
Solving puzzles: Sudoku and more

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Sudoku 数独 / すうどく

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

You are given a 9-by-9 grid with 81 cells. Certain cells are pre-filled with numbers. Your goal is to complete this grid such that each column, each row, and each of the nine 3x3 subgrids that compose the grid contain all of the digits from 1 to 9.

Constraint satisfaction problem

- In Sudoku, there is no preference among feasible solutions. That is, all feasible solution are equally good, there is no on is "better" than the others.
- These problems are called Constraint satisfaction problem (CSP): you just want to find (any) one of the (basic) feasible solution that satisfy the constraint.
- In terms of geometry: if the feasible region is non-empty and contains at least one feasible solution: the CSP problem is solvable.
- In CSP, since we do not have preference, hence the objective function is zero.

$$\begin{array}{ll} \min\limits_{\mathbf{x}} & 0 \\ \text{s.t.} & \text{constraints.} \end{array}$$

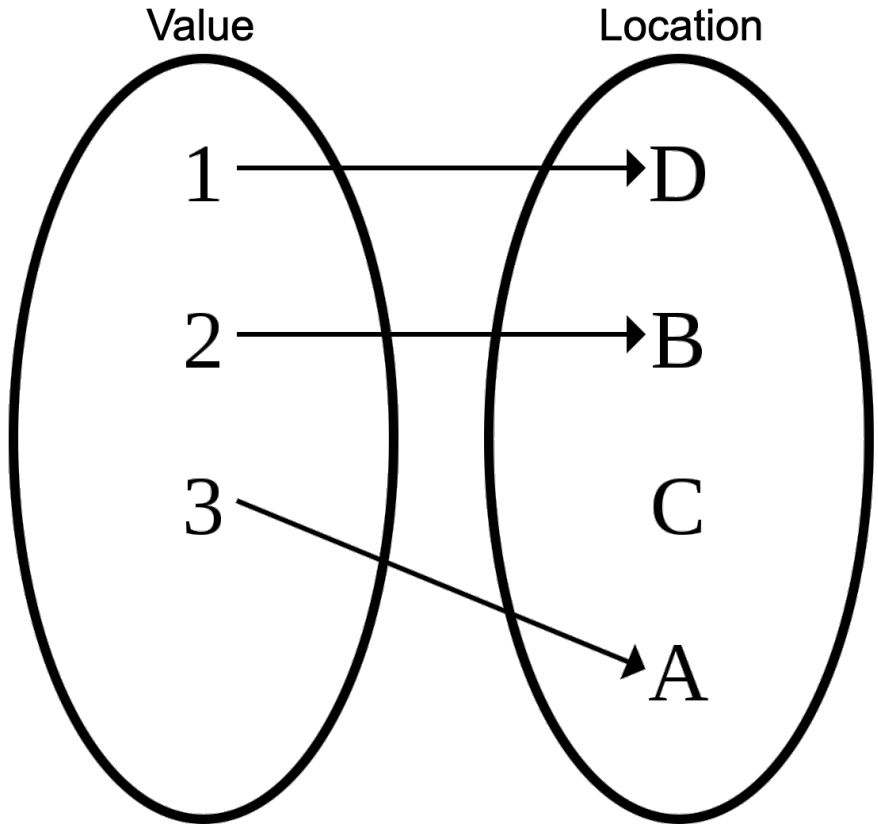
- Hence in modeling Sudoku (and also N-Queen), we mainly need to focus on modeling the constraint.

Constraint in Sudoku

- Values can only be 1-9.
- Only one value is contained with each column.
  i.e., no repeated value in column.
- Only one value is contained with a row.
  i.e., no repeated value in row.
- Only one value is contained with a each 3-by-3 subgrid. i.e., no repeated value in subgrid.

Decision variable ... (1/2)

- How to determine the decision variable?
- What we are ask to do: place numbers in each specific box
  - Numbers 1-9: the first set
  - Box location: second set



- So first we can let

$$x_{\text{num,loc}} = 1 \iff \text{place value “num” in location “loc”}$$

Decision variable ... (2/2)

- In fact, the order of the relation is important:
  - We are given a list of location
  - Our goal is to place number in these locations.

  Therefore the decision variable should be

$$x_{\text{loc,num}}$$

- What is "location": xy-coordinate.
- What is "number" : values in 1-9,
- Therefore we can define the decision variable as x_{ijk},

$$x_{ijk} = 1 \iff \text{place value } k \text{ in location } (i, j)$$

- The range of i, j, k is then 0 - 9.

Deciding the decision variable

- The key is to identify what the binary variable is doing: linking two sets.
- When deciding the decision variable, identifying the two sets is most important step
- This step can be nontrivial and it is easy to make mistake!

Modeling the constraint ... (1/2)

- Only one value is contained with each column.

$$\sum_{i=1}^{9} x_{ijk} = 1, \text{ for all } j, k = 1, ..., 9$$

- Only one value is contained with a row.

$$\sum_{j=1}^{9} x_{ijk} = 1, \text{ for all } i, k = 1, ..., 9$$

- Only one value is contained with a each 3-by-3 subgrid.

$$\sum_{i=3p-2}^{3p} \sum_{j=3q-2}^{3q} x_{ijk} = 1, \text{ for all } k = 1, ..., 9, \text{ and } p, q = 1, ..., 3$$

Modeling the constraint ... (2/2)

- You can only place at most 1 number in each box

$$\sum_{k=1}^{9} x_{ijk} = 1, \text{ for all } i, j = 1, ..., 9$$

  This one is easy to miss!

- Some box is pre-filled.

$$x_{ijk} = 1, \text{ for all } (i, j, k) \in G := \text{all known cells.}$$

The integer program of Sudoku

$$\min \quad 0$$

$$\text{s.t.} \quad \sum_{i=1}^{9} x_{ijk} = 1, \text{ for } j,k = 1,...,9$$

$$\sum_{j=1}^{9} x_{ijk} = 1, \text{ for } i,k = 1,...,9$$

$$\sum_{i=3p-2}^{3p} \sum_{j=3q-2}^{3q} x_{ijk} = 1, \text{ for } k = 1,...,9, \text{ and } p,q = 1,...,3$$

$$\sum_{k=1}^{9} x_{ijk} = 1, \text{ for } i,j = 1,...,9$$

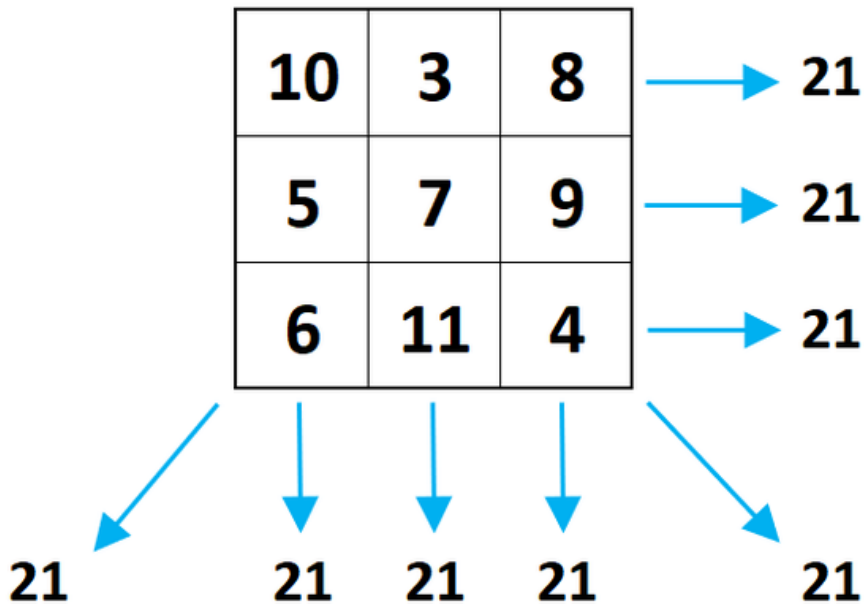
$$x_{ijk} = 1, \text{ for all } (i,j,k) \in G := \text{all known cells.}$$

About the IP

- x_{ijk} for $i, j, k = 1, ..., 9$, There are in total $9 \times 9 \times 9 = 729$ variables.
- How many constraints?
  $\rightarrow$ assignment.
- Can we model Sudoku using IP instead of BIP?

Magic square

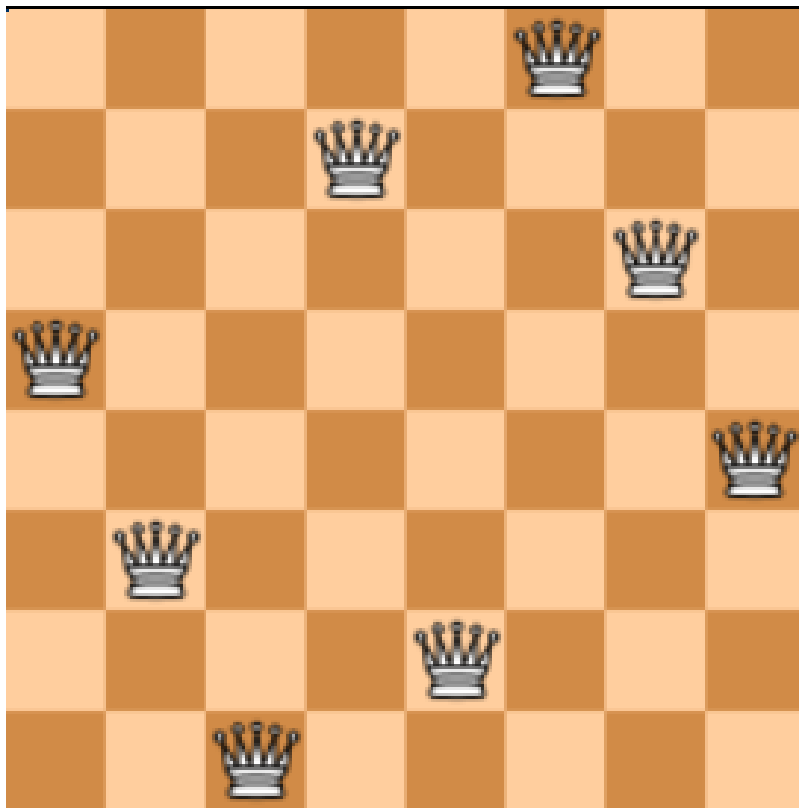
- Given a n-by-n array.
- Magic square $=$ fill this array by distinct positive integers in $1, \ldots, M$ such that the sum of the n numbers in any row, column, main diagonal is always the same number.



- M is not necessarily $= n^2$.
- Formulate this as a IP (assignment)

N-Queen

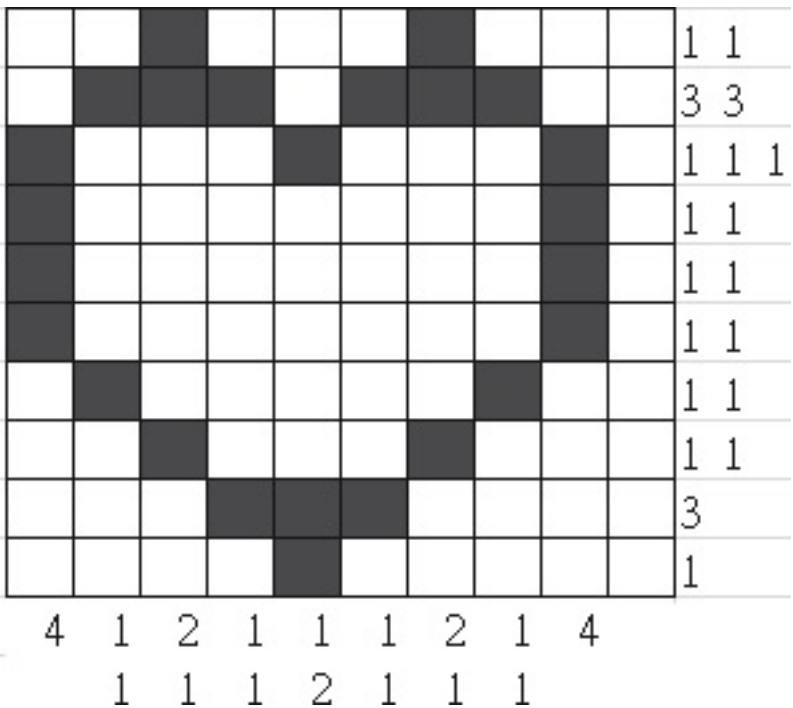
- Given a N-by-N chess board. Place N queens on the board such that no two queens "see each other":
  - They cannot be on the same row
  - They cannot be on the same column
  - They cannot be on the same diagonal

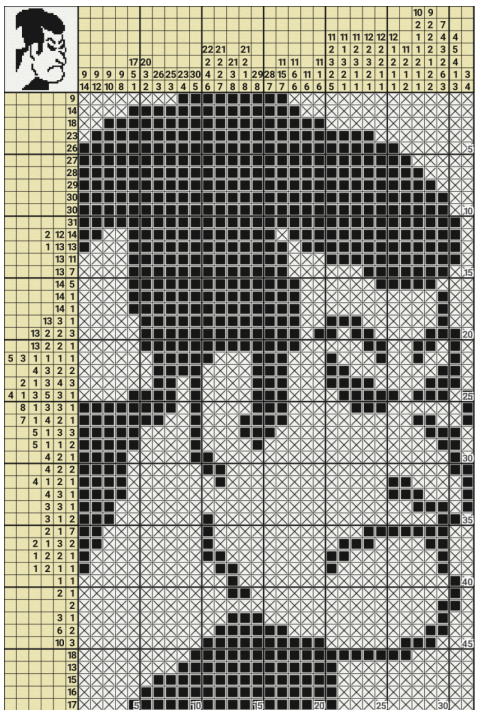


- Formulate this problem as a IP. (assignment)

Nonogram ののぐらむ

- By Japanese Non Ishida and Tetsuya Nishio.
- Given: a grid of skyscraper lights with a set of integer strings
- Goal: turned on or off the skyscraper lights to form an image according to the integer strings





- Formulate this problem as a IP (assignment?).