CO327 Deterministic OR Models (2022-Spring)

Illustration Fourier-Motzkin Elimination

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Solving system of linear inequalities

- You probably know how to solve $\mathbf{Ax} = \mathbf{b}$. What about solving $\mathbf{Ax} \leq \mathbf{b}$?
- Determine if the following system of inequalities has a solution:

$$\mathbf{Ax} \leq \mathbf{b} \quad \Longleftrightarrow \quad \begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \leq & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \leq & b_2 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \leq & b_m \end{array}$$

- Method: Fourier-Motzkin elimination.
  Two general steps:
  - Normalization
  - Elimination

Example 1

- Determine if the following system of inequalities has a solution:

$$\mathcal{P} \; : \; \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} \in \mathbb{R}^2 \; : \; \begin{cases} x_1 + x_2 & \geq & 0 \\ 2x_1 + x_2 & \geq & 2 \\ -x_1 + x_2 & \geq & 1 \\ -x_1 + 2x_2 & \geq & -1 \end{cases}$$

  That is, is the set $\mathcal{P}$ contains one feasible point.

- The system has a solution $\iff$ $\mathcal{P}$ contains (at least) one point.
- The system has no solution $\iff$ $\mathcal{P}$ is a empty set.

Example 1. Normalization step

- Make x_1 as the subject:

$$
\mathcal{P} \iff \begin{array}{rclcrcll}
x_1 & \geq & -x_2 & & & & & (1) \\
x_1 & \geq & 1 - \frac{1}{2}x_2 & & & & & (2) \\
-x_1 & \geq & 1 - x_2 & \iff & x_1 & \leq & -1 + x_2 & (3) \\
-x_1 & \geq & -1 - 2x_2 & \iff & x_1 & \leq & 1 + 2x_2 & (4)
\end{array}
$$

- Idea: $\mathcal{P}$ is equivalent to (1),(2),(3),(4) because what we have performed are elementary operation (the one you learn from linear algebra).

- $\mathcal{P}$ has a point $\iff$ (1),(2),(3),(4) are all true.

- (1),(2) and (3),(4) on x_1 are on opposite sign, we can combine them to get an expression in the form of

$$\text{a function of } x_2 \leq x_1 \leq \text{ another function of } x_2$$

Example 1. Elimination step

$$\mathcal{P} \iff \begin{array}{rcllcll} x_1 & \geq & -x_2 & & & & (1) \\ x_1 & \geq & 1 - \frac{1}{2}x_2 & & & & (2) \\ -x_1 & \geq & 1 - x_2 & \iff & x_1 & \leq & -1 + x_2 \quad (3) \\ -x_1 & \geq & -1 - 2x_2 & \iff & x_1 & \leq & 1 + 2x_2 \quad (4) \end{array}$$

- Create a new system of linear inequalities

$$\mathcal{P}' : \begin{array}{rcll} -x_2 & \leq x_1 \leq & -1 + x_2 & (1)\&(3) \\ -x_2 & \leq x_1 \leq & 1 + 2x_2 & (1)\&(4) \\ 1 - \frac{1}{2}x_2 & \leq x_1 \leq & -1 + x_2 & (2)\&(3) \\ 1 - \frac{1}{2}x_2 & \leq x_1 \leq & 1 + 2x_2 & (2)\&(4) \end{array}$$

- Note that the new system is equivalent to the old system, but it has one variable less than the old system: ignore the middle term gives

$$\begin{array}{rcll} -x_2 & \leq & -1 + x_2 & (1)\&(3) \\ -x_2 & \leq & 1 + 2x_2 & (1)\&(4) \\ 1 - \frac{1}{2}x_2 & \leq & -1 + x_2 & (2)\&(3) \\ 1 - \frac{1}{2}x_2 & \leq & 1 + 2x_2 & (2)\&(4) \end{array}$$

Example 1. proceed to the next iteration

- Now we have

$$
\begin{array}{rcll}
-x_2 & \leq & -1 + x_2 & (1)\&(3) \\
-x_2 & \leq & 1 + 2x_2 & (1)\&(4) \\
1 - \frac{1}{2}x_2 & \leq & -1 + x_2 & (2)\&(3) \\
1 - \frac{1}{2}x_2 & \leq & 1 + 2x_2 & (2)\&(4)
\end{array}
$$

- Simply the new system gives

$$
\begin{array}{rcl}
\frac{1}{2} & \leq & x_2 \\
-\frac{1}{3} & \leq & x_2 \\
\frac{4}{3} & \leq & x_2 \\
0 & \leq & x_2
\end{array}
$$

  which gives $x_2 \geq \frac{4}{3}$.

- Hence the original $\mathcal{P}$ has a solution if and only if $x_2 \geq \frac{4}{3}$.

Example 1. solution

- Now we know $x_2 \geq \frac{4}{3}$, what about x_1?
- To get x_1, reuse $\mathcal{P}'$

$$\mathcal{P}' \; : \; \begin{array}{rcll} -x_2 & \leq x_1 \leq & -1 + x_2 & (1)\&(3) \\ -x_2 & \leq x_1 \leq & 1 + 2x_2 & (1)\&(4) \\ 1 - \frac{1}{2}x_2 & \leq x_1 \leq & -1 + x_2 & (2)\&(3) \\ 1 - \frac{1}{2}x_2 & \leq x_1 \leq & 1 + 2x_2 & (2)\&(4) \end{array}$$

- Simply pick $x_2 = 2$ gives

$$\begin{array}{rcll} -2 & \leq x_1 \leq & 1 & (1)\&(3) \\ -2 & \leq x_1 \leq & 5 & (1)\&(4) \\ 0 & \leq x_1 \leq & 1 & (2)\&(3) \\ 0 & \leq x_1 \leq & 5 & (2)\&(4) \end{array}$$

- Take the intersection of these set gives $0 \leq x_1 \leq 1$.
- So *a* solution to $\mathcal{P}$ is $(0, 2)$ and there are infinitely many solution to $\mathcal{P}$.

Exercise question

- Determine if the following system of inequalities has a solution:

$$\begin{aligned} x_1 + x_2 - 2x_3 &\geq 2 \\ -x_1 - 3x_2 + x_3 &\geq 1 \\ x_2 + x_3 &\geq 0 \end{aligned}$$

A solution of the practice problem.

- Step 1. Eliminate x_1.

$$
\begin{array}{rrcll}
 & x_1 + x_2 - 2x_3 & \geq & 2 & (1) \\
 & -x_1 - 3x_2 + x_3 & \geq & 1 & (2) \\
 & x_2 + x_3 & \geq & 1 & (3) \\
(1)+(2) & -2x_2 - x_3 & \geq & \not{2}\ 3 & \\
\iff & -x_2 - \frac{1}{2}x_3 & \geq & 1 & (4)
\end{array}
$$

- Step 2. Combine (4) and (3) gives $\frac{1}{2}x_3 \geq 2$, which is $x_3 \geq 4$.
- Set $x_3 = 4$, then (4) gives $x_2 \leq -3$.
- Set $x_2 = -3$, then (1) gives $x_1 \geq 13$.
- So *a* solution is ~~$(4, -3, 13)$~~. $(13, -3, 4)$
- Since there are infinitely many solution so here is one particular solution.
- If your solution is different, you can verify the solution by checking all the inequalities.

Assignment

- Determine if the following system of inequalities has a solution:

$$\begin{aligned} x_1 + x_2 + 2x_3 &\geq 1 \\ -x_1 + x_2 + x_3 &\geq 2 \\ x_1 - x_2 + x_3 &\geq 1 \\ -x_2 - 3x_3 &\geq 0 \end{aligned}$$

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