CO327 Deterministic OR Models (2022-Spring)

IP modelling tricks: big-M, or condition and if-then condition **updated**

Instructor: Andersen Ang

Combinatorics and Optimization, U.Waterloo, Canada

msxang@uwaterloo.ca, where $\mathbf{x} = \lfloor \pi \rfloor$

Homepage: angms.science

First draft: June 2, 2022 Last update: June 6, 2022

1. The big-M trick: a technique to turn constraint on/off
2. The or condition
3. The if-then condition

A production model

- Gandhi Cloth Company is capable of manufacturing three types of clothing: shirts, shorts, and pants. The manufacture of each type of clothing requires that Gandhi have the appropriate type of machinery available. The machinery needed to manufacture each type of clothing must be rented at the following rates: shirt machinery, \$200 per week; shorts machinery, \$150 per week; pants machinery, \$100 per week. The manufacture of each type of clothing also requires the amounts of cloth and labor shown in Table 2. Each week, 150 hours of labor and 160 sq yd of cloth are available. The variable unit cost and selling price for each type of clothing are shown in Table 3. Formulate an IP whose solution will maximize Gandhi's weekly profits.

TABLE 2

Resource Requirements for Gandhi

Clothing Type	Labor (Hours)	Cloth (Square Yards)
Shirt	3	4
Shorts	2	3
Pants	6	4

TABLE 3

Revenue and Cost Information for Gandhi

Clothing Type	Sales Price (\$)	Variable Cost (\$)
Shirt	12	6
Shorts	8	4
Pants	15	8

A production model : variables

- Gandhi Cloth Company is capable of manufacturing three types of clothing: shirts, shorts, and pants.

$$\begin{aligned} x_1 &= \text{number of shirts produced each week} \\ x_2 &= \text{number of shorts produced each week} \\ x_3 &= \text{number of pants produced each week} \end{aligned}$$

- The manufacture of each type of clothing requires that Gandhi have the appropriate type of machinery available.

$$y_1 = \begin{cases} 1 & \text{if any shirts are manufactured} \\ 0 & \text{else} \end{cases}$$

$$y_2 = \begin{cases} 1 & \text{if any shorts are manufactured} \\ 0 & \text{else} \end{cases}$$

$$y_3 = \begin{cases} 1 & \text{if any pants are manufactured} \\ 0 & \text{else} \end{cases}$$

Cost function/Objective function

- The machinery needed to manufacture each type of clothing must be rented at the following rates: shirt machinery, \$200 per week; shorts machinery, \$150 per week; pants machinery, \$100 per week.

  Weekly cost of renting machinery

$$200y_1 + 150y_2 + 100y_3$$

- Weekly net profits of selling clothing

TABLE 3

Revenue and Cost Information for Gandhi

Clothing Type	Sales Price (\$)	Variable Cost (\$)
Shirt	12	6
Shorts	8	4
Pants	15	8

$$\begin{aligned} & (12x_1 + 8x_2 + 15x_3) - (6x_1 + 4x_2 + 8x_3) \\ = \quad & 6x_1 + 4x_2 + 7x_3 \end{aligned}$$

- Total weekly profit

$$\max_{x_1, x_2, x_3, y_1, y_2, y_3} \quad 6x_1 + 4x_2 + 7x_3 - 200y_1 + 150y_2 + 100y_3$$

Constraints

- At most 150 hours of labor can be used each week.

$$3x_1 + 2x_2 + 6x_3 \leq 150$$

- At most 160 sq yd of cloth can be used each week

$$4x_1 + 3x_2 + 4x_3 \leq 160$$

- Nonnegative production

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0$$

- Integer amount of clothing

$$x_1 \in \mathbb{N}, \quad x_2 \in \mathbb{N}, \quad x_3 \in \mathbb{N}$$

- Binary decision of renting machine

$$y_1 \in \{0, 1\}, \quad y_2 \in \{0, 1\}, \quad y_3 \in \{0, 1\}$$

The optimization problem

- Putting everything together

$$
\begin{array}{rl}
\max\limits_{x_1,x_2,x_3,y_1,y_2,y_3} & 6x_1 + 4x_2 + 7x_3 - 200y_1 + 150y_2 + 100y_3 \\
\text{s.t.} & 3x_1 + 2x_2 + 6x_3 \leq 150 \\
& 4x_1 + 3x_2 + 4x_3 \leq 160 \\
& x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0, x_1 \in \mathbb{N}, x_2 \in \mathbb{N}, x_3 \in \mathbb{N} \\
& y_1 \in \{0,1\}, y_2 \in \{0,1\}, y_3 \in \{0,1\}
\end{array}
$$

- This model is actually wrong. The optimal solution:

$$x_1 = 30, \quad x_2 = 0, \quad x_3 = 10, \quad y_1 = y_2 = y_3 = 0.$$

  Meaning: Gandhi does not rent any machine and is still able to produce clothing.

- What is missing here

$$x_i > 0 \implies y_i = 1$$

  Meaning: if Gandhi produces pants, he must rent the pant machine.

The big-M trick

- Let M be a very large number. The constraints
$$\begin{aligned} x_1 &\leq My_1 \\ x_2 &\leq My_2 \\ x_3 &\leq My_3 \end{aligned}$$
solve the modelling error.

- Consider $x_1 \leq My_1$
  - If $y_1 = 0$ (Gandhi does not rent the shirt machine)
    - We have $x_1 \leq M \cdot 0 = 0$, that is, $x_1 \leq 0$.
    - As we have $x_1 \geq 0$ in the optimization, then $x_1 \geq 0$ AND $x_1 \leq 0$ implies $x_1 = 0$.

    In conclusion:
$$\text{if } y_1 = 0 \text{ (no shirt machine)} \quad \Longrightarrow \quad x_1 = 0 \text{ (no shirt produced)}$$

  - If $y_1 = 1$ (Gandhi rents the shirt machine)
    - We have $x_1 \leq M \cdot 1 = M$, that is, $x_1 \leq M$.
    - As we have $x_1 \geq 0$ in the optimization, then $x_1 \geq 0$ AND $x_1 \leq M$ implies $0 \leq x_1 \leq M$.
    - M is big $\implies$ $x_1 \leq M$ is "approximately" $x_1 \leq +\infty$, meaning almost no upper bound on x_1

    In conclusion:
$$\text{if } y_1 = 1 \text{ (rent shirt machine)} \quad \Longrightarrow \quad 0 \leq x_1 \leq M \quad \approxeq \quad 0 \leq x_1 \text{ (make non-negative amount of shirt)}$$

The (correct) optimization problem

- Putting everything together

$$
\begin{aligned}
\max_{x_1,x_2,x_3,y_1,y_2,y_3} \quad & 6x_1 + 4x_2 + 7x_3 - 200y_1 + 150y_2 + 100y_3 \\
\text{s.t.} \quad & 3x_1 + 2x_2 + 6x_3 \le 150 \\
& 4x_1 + 3x_2 + 4x_3 \le 160 \\
& x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0, x_1 \in \mathbb{N}, x_2 \in \mathbb{N}, x_3 \in \mathbb{N} \\
& y_1 \in \{0,1\}, y_2 \in \{0,1\}, y_3 \in \{0,1\} \\
& x_1 \le M y_1 \\
& x_2 \le M y_2 \\
& x_3 \le M y_3
\end{aligned}
$$

- The optimal solution: $x_3 = 25, y_3 = 1$. Gandhi should produce 25 pants each week.

IMPORTANT: The big-M trick, summary

$$0 \le x \le My, \quad M \to +\infty, \quad y \in \{0,1\}$$

$y = 0$	$y = 1$
$\downarrow$	$\downarrow$
$0 \le x \le 0$	$0 \le x \le M$
$\downarrow$	$\downarrow$ approximately
$x = 0$	$0 \le x$

In words: if x is a continuous variable and y is a binary variable, then the constraint

$$0 \le x \le My$$

says that if $y = 0$, then x must equal 0, and if $y = 1$, then the constraint places (almost) no restrictions on x. The binary variable y "turns" constraints on or off.

Another production model

- Dorian Auto is considering manufacturing three types of autos: compact, midsize, and large. The resources required for, and the profits yielded by, each type of car are shown in Table 8. Currently, 6000 tons of steel and 60000 hours of labor are available. For production of a type of car to be economically feasible, at least 1000 cars of that type must be produced. Formulate an IP to maximize Dorian's profit.

TABLE 8

Resources and Profits for Three Types of Cars

	Car Type		
Resource	**Compact**	**Midsize**	**Large**
Steel required	1.5 tons	3 tons	5 tons
Labor required	30 hours	25 hours	40 hours
Profit yielded ($)	2,000	3,000	4,000

Modelling the problem

- Dorian Auto is considering manufacturing three types of autos: compact, midsize, and large.

$$\begin{aligned} x_1 &= \text{number of compact cars produced} \\ x_2 &= \text{number of midsize cars produced} \\ x_3 &= \text{number of large cars produced} \end{aligned}$$

- The profit (in thousands of dollars)

$$\max_{x_1, x_2, x_3} 2000x_1 + 3000x_2 + 4000x_3$$

- Easy constraints
  - The cars produced can use at most 6000 tons of steel.
  - The cars produced can use at most 60000 hours of labor.
  - Nonnegative production.
  - Integer number of cars.

The or statement

- For production of a type of car to be economically feasible, at least 1000 cars of that type must be produced.

$$
\begin{aligned}
x_1 &= 0 \quad \text{or} \quad x_1 \geq 1000 \\
x_2 &= 0 \quad \text{or} \quad x_2 \geq 1000 \\
x_3 &= 0 \quad \text{or} \quad x_3 \geq 1000
\end{aligned}
$$

- Putting things together

$$
\begin{array}{rll}
\max\limits_{x_1,x_2,x_3} & 2000x_1 + 3000x_2 + 4000x_3 & \\
\text{s.t.} & 1.5x_1 + 3x_2 + 5x_3 \leq 6000 & \text{(steel constraint)} \\
& 30x_1 + 25x_2 + 40x_3 \leq 60000 & \text{(labour constraint)} \\
& x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 & \text{(non-negative production)} \\
& x_1 \in \mathbb{N}, x_2 \in \mathbb{N}, x_3 \in \mathbb{N} & \text{(integral number of cars)} \\
& x_1 = 0 \text{ or } x_1 \geq 1000 & \text{(economic constraint of compact car)} \\
& x_2 = 0 \text{ or } x_2 \geq 1000 & \text{(economic constraint of midsize car)} \\
& x_3 = 0 \text{ or } x_3 \geq 1000 & \text{(economic constraint of large car)}
\end{array}
$$

- How to deal with the or statement?????

The OR logic

Suppose we have "A or B"

A	B	A inclusive-or B	A exclusive-or B
T	T	T	F
T	F	T	T
F	T	T	T
F	F	F	F

Examples

- "Peter or David speaks French."
  - Peter speaks French.
  - David speaks French.
  - Both Peter and David speak French.
- "The salad comes with bacon or sausage."
  - Salad with bacon.
  - Salad with sausage.
  - Salad with bacon and sausage? No way.

How to deal with the inclusive-or statement

- Given two constraints

$$f(x_1, x_2, \ldots, x_n) \le 0, \qquad g(x_1, x_2, \ldots, x_n) \le 0.$$

  We want: at least one of them is satisfied (in the inclusive-or sense).

- The conversion

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &\le My \\ g(x_1, x_2, \ldots, x_n) &\le M(1-y) \\ y &\in \{0, 1\} \end{aligned}$$

$y = 0$	$y = 1$
$\downarrow$	$\downarrow$
$f \le 0, \quad g \le M$	$f \le M, \quad g \le 0$
$\downarrow$	$\downarrow$
$f \le 0$ true & $g \le 0$ don't know (possibly true)	$g \le 0$ true & $f \le 0$ don't know (possibly true)

Back to the auto problem

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &\leq My \\ g(x_1, x_2, \ldots, x_n) &\leq M(1-y) \\ y &\in \{0,1\} \end{aligned} \quad \Longleftrightarrow \quad f(x_1, x_2, \ldots, x_n) \leq 0 \text{ or } g(x_1, x_2, \ldots, x_n) \leq 0$$

On the auto problem:

- $x_1 = 0$ or $x_1 \geq 1000$ is equivalent to

$$\begin{aligned} x_1 &\leq M_1 y_1 \\ 1000 - x_1 &\leq M_1(1 - y_1) \\ y_1 &\in \{0,1\} \end{aligned}$$

- How:
  - With $x_1 \geq 0$, we know that

$$x_1 = 0 \iff \underbrace{x_1 \geq 0}_{\text{already in the problem}} \quad \text{AND} \quad \underbrace{x_1 \leq 0}_{\text{our } f(x) \leq 0}$$

  - $x_1 \geq 1000 \iff \underbrace{1000 - x_1 \leq 0}_{\text{our } g(x) \leq 0}$

Explanation

$$\begin{aligned} x_1 &\le M_1 y_1 \\ 1000 - x_1 &\le M_1(1 - y_1) \\ y_1 &\in \{0, 1\} \end{aligned}$$

$$y_1 = 0$$

$$\downarrow$$

$$\begin{aligned} x_1 &\le 0 \\ 1000 - x_1 &\le M_1 \end{aligned}$$

$$\downarrow (x_1 \ge 0)$$

$$x_1 = 0$$

$$\downarrow$$

produce no compact car

$$y_1 = 1$$

$$\downarrow$$

$$\begin{aligned} x_1 &\le M_1 \\ 1000 - x_1 &\le 0 \end{aligned}$$

$$\downarrow$$

$$1000 \le x_1 \le M_1$$

$\downarrow$ approximately

$$1000 \le x_1$$

$$\downarrow$$

produce at least 1000 compact car

The (correct) optimization problem

$$
\begin{array}{lll}
\max\limits_{x_1,x_2,x_3} & 2000x_1 + 3000x_2 + 4000x_3 & \\
\text{s.t.} & 1.5x_1 + 3x_2 + 5x_3 \le 6000 & \text{(steel constraint)} \\
& 30x_1 + 25x_2 + 40x_3 \le 60000 & \text{(labour constraint)} \\
& x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 & \text{(non-negative production)} \\
& x_1 \in \mathbb{N}, x_2 \in \mathbb{N}, x_3 \in \mathbb{N} & \text{(integral number of cars)} \\
& x_1 \le M_1 y_1 & \text{(economic constraint of compact car)} \\
& 1000 - x_1 \le M_1(1 - y_1) & \text{(economic constraint of compact car)} \\
& y_1 \in \{0, 1\} & \text{(economic constraint of compact car)} \\
& x_2 \le M_2 y_2 & \text{(economic constraint of midsize car)} \\
& 1000 - x_2 \le M_2(1 - y_2) & \text{(economic constraint of midsize car)} \\
& y_2 \in \{0, 1\} & \text{(economic constraint of midsize car)} \\
& x_3 \le M_3 y_3 & \text{(economic constraint of large car)} \\
& 1000 - x_3 \le M_3(1 - y_3) & \text{(economic constraint of large car)} \\
& y_3 \in \{0, 1\} & \text{(economic constraint of large car)}
\end{array}
$$

- What are M_i, just pick a big number, for example

$$M_1 = M_2 = M_3 = 2000$$

- Optimal sol $x_2 = 2000, y_2 = 1, y_1 = y_3 = x_1 = x_3 = 0$. Dorian should produce 2000 midsize cars.

IMPORTANT: inclusive OR, summary

$$f(x_1, x_2, \ldots, x_n) \le 0 \ \text{ OR } \ g(x_1, x_2, \ldots, x_n) \le 0 \quad \Longleftrightarrow \quad \begin{aligned} f(x_1, x_2, \ldots, x_n) &\le My \\ g(x_1, x_2, \ldots, x_n) &\le M(1-y) \\ y &\in \{0, 1\} \end{aligned}$$

$y = 0$	$y = 1$
$\downarrow$	$\downarrow$
$f \le 0, \quad g \le M$	$f \le M, \quad g \le 0$
$\downarrow$	$\downarrow$
$f \le 0$ turn on ($g \le 0$ don't know)	$g \le 0$ turn on ($f \le 0$ don't know)

Note: inclusive or means that both $f \le 0$ and $g \ge 0$ can hold at the same time.

How to deal with the if-then statement

- Given two constraints

$$f(x_1, x_2, \ldots, x_n) > 0, \qquad g(x_1, x_2, \ldots, x_n) \geq 0.$$

- We want: ensure if $f > 0$ is satisfied then $g \geq 0$ is satisfied

$$\begin{aligned} -g(x_1, x_2, \ldots, x_n) &\leq My \\ f(x_1, x_2, \ldots, x_n) &\leq M(1-y) \\ y &\in \{0, 1\} \end{aligned}$$

---

if $f > 0$ is true	if $f > 0$ is false
$\downarrow$	$\downarrow$
$y = 0$	y can be 0 or 1
$\downarrow$	$\downarrow$
$-g \leq 0 \iff g \geq 0$	don't know what will happen to g

Fire station example

- Consider location problem with $x_i \in \{0,1\}$ as

$$\begin{aligned} x_1 &= \text{1 means place a fire station in city 1 and } = \text{0 means not} \\ x_2 &= \text{1 means place a fire station in city 2 and } = \text{0 means not} \\ x_3 &= \text{1 means place a fire station in city 3 and } = \text{0 means not} \end{aligned}$$

- "You place exactly one fire station among cities 1,2,3"

$$x_1 + x_2 + x_3 = 1.$$

Fire station example

- "If you place a fire station in city 1, then no other cities may build a fire station "

$$\text{if } x_1 = 1, \qquad \text{then } x_2 = x_3 = 0 \tag{$*$}$$

  - If $x_1 = 1$ is true, then we know x_2, x_3 must both be zero
    The information "$x_1 = 1$ is true" gives us the conclusion $x_2 = x_3 = 0$
  - If $x_1 = 1$ is false, then we know nothing: x_2, x_3 can be anything.
    "$x_1 = 1$ is false" does not give us enough information to make any conclusion here

The tricky step

$$\text{if } x_1 = 1, \qquad \text{then } x_2 = x_3 = 0 \tag{$*$}$$

$$\Updownarrow$$

$$\underbrace{\text{if } x_1 > 0, \qquad \text{then } x_2 + x_3 \le 0}_{A} \quad \textbf{OR} \quad \underbrace{-x_2 - x_3 \ge 0}_{B} \tag{\#}$$

- $f = x_1$, $g = -x_2 - x_3$, apply the or trick here: (#) is equivalent to

$$\begin{aligned} -(-x_2 - x_3) &\le My \\ x_1 &\le M(1-y) \\ y &\in \{0,1\} \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} x_2 + x_3 &\le My \\ x_1 &\le M(1-y) \\ y &\in \{0,1\} \end{aligned}$$

- In other words,

"If you place a fire station in city 1, then no other cities may build a fire station "

$$\Updownarrow$$

$$\begin{aligned} x_2 + x_3 &\le My \\ x_1 &\le M(1-y) \\ y &\in \{0,1\} \end{aligned}$$

Explanation

$$\begin{aligned} x_2 + x_3 &\leq My \\ x_1 &\leq M(1-y) \\ y &\in \{0,1\} \end{aligned}$$

Note: instead of considering $y = 1$ or $y = 0$, here we look at x_1

$$x_1 = 1$$

$$\downarrow$$

$$\begin{aligned} y &= 0 \\ x_2 + x_3 &\leq 0 \end{aligned}$$

$\downarrow$ (we have $x_2, x_3 \geq 0$)

$$x_2 = x_3 = 0, \quad x_1 = 1 \leq M, y = 0$$

$$x_1 = 0$$

$$\downarrow$$

you don't know what will happen as

y can be 0 or 1

$$\downarrow$$

end of story

More example: if $z = 0$ then $x = y$

$$\begin{aligned} x - y &\leq Mz \\ x - y &\geq -Mz \\ z &\in \{0, 1\} \end{aligned}$$

$z = 1$	$z = 0$
$\downarrow$	$\downarrow$
$\begin{aligned} x - y &\leq M \\ x - y &\geq -M \end{aligned}$	$\begin{aligned} x - y &\leq 0 \\ x - y &\geq 0 \end{aligned}$
$\downarrow$	$\downarrow$
$-M \leq x - y \leq M$	$0 \leq x - y \leq 0$
$\downarrow$ (approximately)	$\downarrow$
$-\infty \leq x - y \leq +\infty$	$x = y$

Practise problem 1

Remove the OR in the following MIP (the or is inclusive or)

$$\begin{aligned} \max_{x_1,x_2} \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 \geq 0, x_2 \geq 0 \\ & x_1 \in \mathbb{N}, x_2 \in \mathbb{N} \\ & x_1 + x_2 \leq 3 \ \ \text{OR} \ \ 2x_1 - x_2 \leq 2 \end{aligned}$$

Solution next page.

Solution

$$
\begin{aligned}
\max_{x_1,x_2} \quad & 2x_1 + 3x_2 \\
\text{s.t.} \quad & x_1 \geq 0, x_2 \geq 0 \\
& x_1 \in \mathbb{N}, x_2 \in \mathbb{N} \\
& x_1 + x_2 \leq 3 + My \\
& 2x_1 - x_2 \leq 2 + M(1-y) \\
& y \in \{0,1\}
\end{aligned}
$$

$y = 1$	$y = 0$
$\downarrow$	$\downarrow$
$x_1 + x_2 \leq 3 + M$ $2x_1 - x_2 \leq 2$	$x_1 + x_2 \leq 3$ $2x_1 - x_2 \leq 2 + M$
$\downarrow$	$\downarrow$
"turn off" $x_1 + x_2 \leq 3$, keep $2x_1 - x_2 \leq 2$	"turn off" $2x_1 - x_2 \leq 2$, keep $x_1 + x_2 \leq 3$

One more example (mid-term question)

- You buy apple and orange for vitamin C.

Fruit	Apple	Orange
Vitamin C content	5 per each	10 per each
Normal price	2$ per each	3.5$ per each
Special deal	1.5$ per each if buy more than 5	10$ for 3 oranges

Table: Information of the apple and oranges

- Examples: for 80 vitamins
  - If you buy 4 apples and 6 oranges (two orange bundles), you will have $5 \times 4 + 10 \times 6 = 80$ vitamin and they cost $2 \times 4 + 10 \times 2 = 28\$$.
  - If you buy 8 apples and 4 oranges (one orange bundle and one orange), you will have $5 \times 8 + 10 \times 4 = 80$ vitamin and they cost $1.5 \times 8 + 10 \times 1 + 3.5 = 25.5\$$.

- Your goal is to spend the minimum amount of money such that the total vitamin C content is at least b amount. Formulate problem as an integer program.

What about other general logic conditions?

- AND
  - Boolean algebra $y = x_1 \wedge x_2$
  - IP
  $$y \geq x_1 + x_2 - 1, \quad y \leq x_1, \quad y \leq x_2, \quad y \in \{0,1\}.$$
  or equivalently
  $$0 \leq x_1 + x_2 - 2y \leq 1.$$
- OR
  - Boolean algebra $y = x_1 \vee x_2$
  - IP
  $$y \leq x_1 + x_2 - 1, \quad y \geq x_1, \quad y \geq x_2, \quad y \in \{0,1\}.$$
  or equivalently
  $$0 \leq 2y - x_1 - x_2 \leq 1.$$
- Logical implication
  - Boolean algebra $y = (x_1 \implies x_2) = \neg x_1 \vee x_2$
  - IP
  $$y \leq 1 - x_1 + x_2, \quad y \geq 1 - x_1, \quad y \geq x_2, \quad y \in \{0,1\}.$$
- Forced logical implication
  - Boolean algebra: forcing $x_1 \implies x_2$ must hold
  - IP
  $$x_1 \leq x_2$$
- XOR
  - Boolean algebra: $y = x_1 \oplus x_2$
  - IP
  $$y \leq x_1 + x_2, \quad y \geq x_1 - x_2, \quad y \geq x_2 - x_1 \quad y \leq 2 - x_1 - x_2, \quad y \in \{0,1\}.$$