Basics about digital image

Andersen Ang
C&O, U Waterloo

CO327 (2021-Spring)

2021-June-11

What is an image

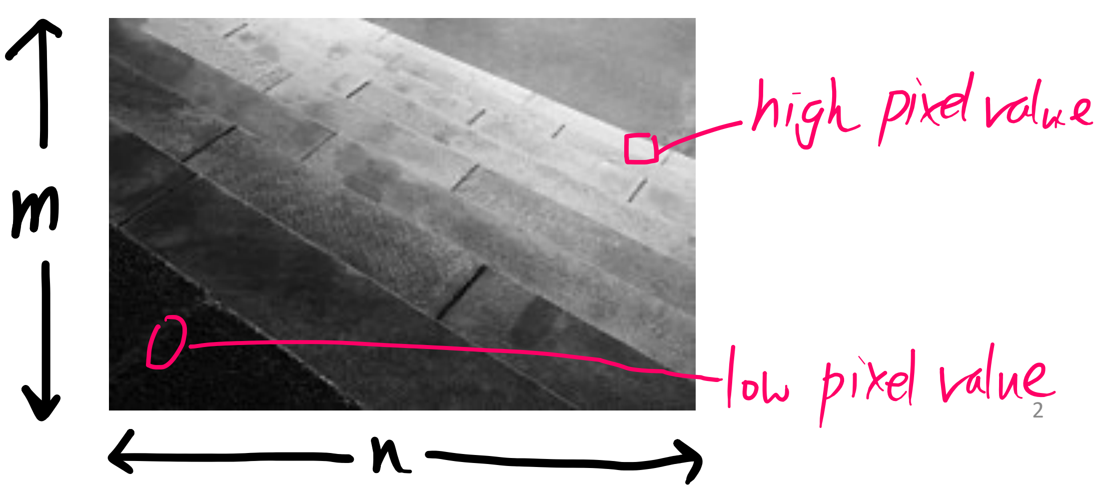
- A matrix of integers
- Each (i,j) element in the matrix is called a "pixel"
- the value of the pixel is the colour value (RGB) or brightness (BW)

Grayscale

- Integer in $[0, 255]$
- 0 = total darkness
- 255 = total whiteness
- 0-255 has $256 = 2^8$ integers

So, what is a grayscale image

- A matrix in $\{0, 1, \ldots, 255\}^{m \times n}$
- A bunch of integers



Application of LP in image inpainting

Andersen Ang, C&O, UWaterloo

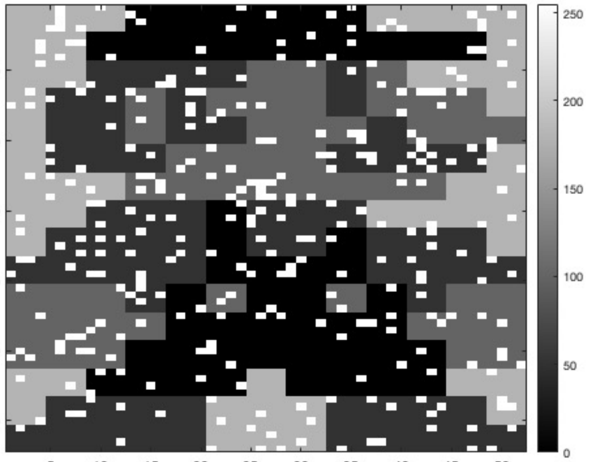
CO327 (2021-spring)

2021-June-8

What is image inpainting

- Restore a "broken" image

Input



Output

The optimization problem of image recovery

$$\min \|Ex\|_1$$
$$\text{s.t. } Sx = \hat{x}_\Omega$$
$$x \geq 0$$

- x is the optimization variable
- $E, S, \hat{x}_\Omega$ are given
- x is the recovered image

Vectorization

- Images are matrices, but we will work on vectors
- Matrix-to-vector transform is called vectorization

$$x = \text{vec}(X)$$

- In MATLAB, it is `x=X(:);`

Illustration of vectorization

$$X = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \in \mathbb{R}^{3\times 3} \xrightarrow{\text{vec}} x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{bmatrix} \in \mathbb{R}^{9}$$

"Stack columns"

x, $\hat{x}_\Omega$ and x_0

- x_0 : the original clean image vector
- $\hat{x}$: the observed broken image vector
- $\hat{x}_\Omega$: the parts of $\hat{x}$ that are clean
- Ω : a set of indicies of clean pixels
- x : the recovered image vector

Example : 2-by-2 case

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & * \\ * & 4 \end{pmatrix} \quad \begin{matrix} 1 & \\ & 4 \end{matrix} \quad \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}$$

X_0	$\hat{X}$	$\hat{X}_\Omega$	X
original image	broken image	clean parts of $\hat{X}$	recovered image (the variable)

$$x_0 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \quad \hat{x} = \begin{pmatrix} 1 \\ * \\ * \\ 4 \end{pmatrix} \quad \hat{x}_\Omega = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad x = \begin{pmatrix} ? \\ ? \\ ? \\ ? \end{pmatrix}$$

Vectorized X_0	$vec(\hat{X})$	$vec(\hat{X}_\Omega)$	$vec(X)$

$$Sx = \hat{x}_\Omega$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 1 \\ 4 \end{pmatrix}}_{\hat{x}_\Omega}$$

$$\Omega = \{1, 4\}$$

- S is a matrix that map x to $\hat{x}_\Omega$
- Meaning of $Sx = \hat{x}_\Omega$: "the recovered image has to match the clean pixels"
- 1st & 4th pixels in x has to match $\hat{x}_\Omega$

$\|Ex\|_1$

- It is called Total variation norm of x
- TV = fluctuation
- Natural image has low fluctuation
- E is a matrix of 1st-order difference

(Optional) What is total variation

- $TV(f) = \int_\Omega \underbrace{\|\nabla f(x)\|_p}\, dx$

"sum" ($\int$), domain of focus (Ω), amount of variation measured by ℓp norm ($\|\nabla f(x)\|_p$)

- TV of vector

$$TV(x) = \underbrace{\sum_{i=2}^{n} |x_{i+1} - x_i| = \left\| \begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \ddots \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} \right\|_1}$$

$$= |x_2 - x_1| + |x_3 - x_2| + \dots + |x_n - x_{n-1}|$$

The optimization problem of image recovery

$$\min_x TV(x)$$ ← recovered image has low TV

$$\text{s.t. } Sx = \hat{x}_\Omega$$ ← recovered image fit the clean pixels

$$x \geq 0$$ ← pixels are nonnegative

- Not LP for the objective function

Turn TV(x) to a linear function

- $TV(x) = \|Ex\|_1$
- $E = \begin{bmatrix} I \otimes D \\ D \otimes I \end{bmatrix}$ is given
- l_1-minimization

$$\left.\begin{aligned} &\min_x \|Ex\|_1 \\ &\text{s.t. } Sx = \hat{x}_\Omega \\ &\quad x \geq 0 \end{aligned}\right\} (*)$$

Your task in assignment ~~4~~ 5

- Given E, S, $\hat{x}_{\Omega}$ and Ω
- Write down a LP for recovery of the broken image
- Run MATLAB to solve the LP
- Show your image

What exactly you need to do

1. Turn (*) in previous page to a LP
2. Write a code to solve this LP
3. Plot the recovered image

See ~~a4~~ a5.pdf for more details.