CO327 Deterministic OR Models (2021-Spring)

A brief introduction to polynomial optimization

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Electric power system

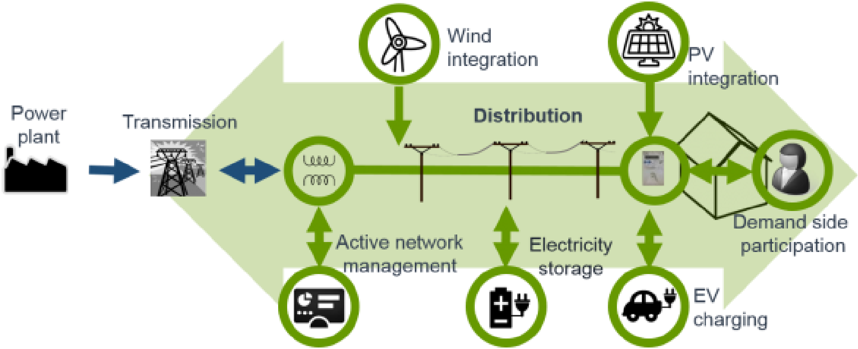


Figure: Image source: `https://www.edsoforsmartgrids.eu`

Optimal power flow problem

- What you have: a power grid (a power network)
  - Power generators: nuclear plant, solar panel, wind turbine
  - Consumers: industry, household, electric vehicles
  - Power transmission lines, power transformers
- What's the goal: minimizing the generation cost
- What's the constraint
  - Physics: Kirchhoff's circuit laws and Ohm's law: you can't violate physics!
  - Engineering: voltages have to be within limit: or the machine explode / overheat.

Network-less economic model

$$\begin{aligned} \min \quad & \sum_i c_i P_{G_i} \\ \text{s.t.} \quad & \sum_i P_{G_i} = \sum_i P_{L_i} \\ & 0 \leq P_{G_i} \leq P_{G_i}^{\max} \end{aligned}$$

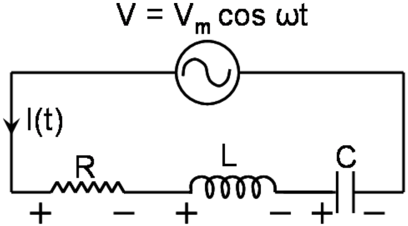
- G_i generator i
- P_{G_i} power of generator i, the decision variable
- c_i generation cost of generator i
- L_i load i
- $P_{G_i}^{\max}$ max power of generator i

The model is network-less: does not consider the physical network.

Network model

- A directed graph: current flow form high voltage to low voltage.
- The problem is to determine how much each generator has to generate electricity with minimum generation cost to fit all demand.
- The problem boils down to computation of voltage in each node.
- V: set of nodes $V = \{1, 2, \ldots, n\}$
- E: set of lines, denoted by $e = (i, j)$
- G_i: set of generators located in node i
- L_i: set of loads located in node i

Some electrical engineering background



- AC current, sinusoids, complex number, $j = \sqrt{-1}$
- Electrical quantity
  - Resistance R, Reactance X and Impedance $Z = R + jX$
  - Conductance G, Susceptance S and Admittance $Y = G + jS$
  - $G = R^{-1}, Y = Z^{-1}, S = X^{-1}$
- We will ignore all technical detail of circuit theory: after balancing the voltage of all node, the resultant problem is a polynomial program.

Polynomial

- A single variable polynomial of degree n

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = \sum_{i=0}^{n} a_i x^i.$$

- A m-variable polynomial of degree n

$$f(\mathbf{x}) = f(x_1, x_2, \dots) = \text{sum of monomial}$$

where a monomial is in the form $x_1^{e_1} \cdot x_2^{e2} \cdots x_m^{e_m}$

Examples

- The number of monomials in a polynomial of m-variable of degree n is

$$\sigma(m,n) = C_m^{m+n} = \frac{(m+n)!}{m!n!}.$$

- Example: 1-variable degree-4 polynomial has $\sigma(1,4) = 5$ monomials:

$$1,\ x,\ x^2,\ x^3,\ x^4.$$

- Example: 2-variable degree-3 polynomial has $\sigma(2,3) = 10$ monomials:

$$1,\ x,\ y,\ x^2,\ xy,\ y^2,\ x^3,\ x^2y,\ xy^2,\ y^3.$$

  Note the *lexicographical ordering* of these monomials.

Polynomial optimization problem example

- Minimize $x_1^2 + x_2^2$
  - Simple quadratic, can be solved by hand.
  - The optimal cost = 0 and the minimizer is $x_1 = x_2 = 0$.

- Minimize $0.33x^3 - x + 1$ subject to $x \geq 0$
  - Not so simple but still "okay".
  - $\frac{d}{dx}(0.33x^3 - x + 1) = 0$ gives minimizer $x \approx 1$.

- Minimize $-0.0071x^5 + 0.087x^4 - 0.1x^3 - 1.3x^2 + 2.3x + 3.2$ subject to $x^3 - 0.03x \leq 0$
  - Use computer.

- Real-world problem: polynomial in 500 variables and 500 constraints.
  - Even experts cannot guarantee they find the global minimum.

Square matrix representation of polynomial

- Example (1-variable degree-2) $f(x) = 1 + x_1 + x_1^2$.

$$f(x) = \begin{bmatrix} 1 \\ x_1 \end{bmatrix}^\top \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \end{bmatrix}^\top .$$

- Example (2-variable degree-3) $f(x) = 7 - 3x_2^2 + 9x_1^3$.

$$f(x) = \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \end{bmatrix}^\top \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4.5 \\ 0 & 0 & -3 & 0 \\ 4.5 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \end{bmatrix} .$$

Square matrix representation of polynomial

- In general, a polynomial $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ can be expressed as the quadratic form $\mathbf{x}_{<n>}{}^\top \mathbf{P} \mathbf{x}_{<n>}$ where $\mathbf{x}_{<n>}$ is the power vector of degree n and $\mathbf{P}$ is a coefficient matrix .
- $\mathbf{x}_{<n>}$ uses the Lexicographical ordering.
- $\mathbf{P}$ can be non-unique.

$$\begin{bmatrix} 1 \\ x_1 \end{bmatrix}^\top \underbrace{\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}}_{\mathbf{P}_1} \begin{bmatrix} 1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ x_1 \end{bmatrix}^\top \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\mathbf{P}_2} \begin{bmatrix} 1 \\ x_1 \end{bmatrix}.$$

- For convenience we stick with $\mathbf{P}$ that is symmetric.

Complete Square matrix representation of polynomial

- If we also take null space into account, the complete matrix representation of polynomial is

$$f(\mathbf{x}) = \mathbf{x}_{<n>}^{\top}\Big(\mathbf{P} + \mathbf{L}\Big)\mathbf{x}_{<n>}$$

where $\mathbf{L}$ is a matrix that $\mathbf{x}_{<n>}^{\top}\mathbf{L}\mathbf{x}_{<n>} = 0$, i.e., $\mathbf{x}_{<n>}$ is inside the null space of $\mathbf{L}$.

- Example (1-variable degree-2)

$$\begin{bmatrix} 1 \\ x_1 \end{bmatrix}^{\top} \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\mathbf{L}} \begin{bmatrix} 1 \\ x_1 \end{bmatrix} = 0 \iff a + (b+c)x + dx^2 = 0.$$

For $a + (b+c)x + dx^2 = 0$ we have each monomial $= 0$, which gives $a = 0$, $b = -c$ and $d = 0$. Let $b = \alpha$, then we have

$$\mathbf{L}(\alpha) = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}.$$

How to solve polynomial optimization

- Cast the cost function into matrix representation form.
- Cast the constraint into matrix representation form (if possible).
- Relax certain nonconvex constraint (if needed).
- Call SDP solver to solve the relaxed problem.
- Check the solution fulfill the constraint or not.
- If the sol. does not fulfill some constraints, add them back

Optimal power flow in real-life is difficult to solve

- Large scale: a country, not a small town with a few houses
  Thousands of variables and constraints.

- Even small improvement can be very significant.

What is U.S. electricity generation by energy source?

In 2020, about 4,009 billion kilowatthours (kWh) (or about 4.01 trillion kWh) of electricity were generated at utility-scale electricity generation facilities in the United States.[1] About 60% of this electricity generation was from fossil fuels—coal, natural gas, petroleum, and other gases. About 20% was from nuclear energy, and about 20% was from renewable

  Saving 0.1% of 1 trillion $=$ 1 billion.

- If you can solve power grid problem efficiently, big company hire you!

Summary

- Brief introduction to polynomial optimization.
- *In this course you will not be tested on solving polynomial optimization problem. The polynomial you will see will be "at most" quadratic function.

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