Solution to CO327 (2021Spring) Assignment 1

Lecturer: Andersen Ang

June 15, 2021

1 Formulation: transportation problem (10 points)

You have a company manufacturing piano. You have 2 factories located at cities F1 and F2 and 3 retail centers located at C1, C2 and C3. The monthly demand at the retail centers are 8, 5 and 2 respectively while the monthly supply at the factories are 6 and 9 respectively. Note that the total supply equals the total demand. You are also given the cost of transportation of 1 piano between any factory and any retail center.

	C1	C2	C3
F1	5	5	3
F2	6	4	1

Table 1: Cost of transportation.

Your goal is to determine the quantity to be transported from each factory to each retail center so as to meet the demand at minimum total shipping cost. Formulate this problem as a linear program/integer program. State clearly your decision variable(s), objective function and constraint(s). *You do not need to solve the program.

1.1 Solution

2 factories and 3 retail centers: 6 decision variable encoding how much each factory ship to each city. Let x_{ij} be shipping from ith city to jth retail center, the decision variables are

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}. \qquad \text{(1 point, 1/6 each)}$$

The objective function is then the total shipping cost

$$5x_{11} + 5x_{12} + 3x_{13} + 6x_{21} + 4x_{22} + 1x_{23} \qquad \text{(1 point, either 1 or 0)}$$

Explicit constraint 1: meet the demand of retail centers

$$\begin{aligned} x_{11} + x_{21} &= 8 \quad \text{meet the demand of retail centers at C1} \\ x_{12} + x_{22} &= 5 \quad \text{meet the demand of retail centers at C2} \\ x_{13} + x_{23} &= 2 \quad \text{meet the demand of retail centers at C3} \end{aligned} \qquad \text{(3 points, 1 point each)}$$

Explicit constraint 2: meet the supply of factories

$$\begin{aligned} x_{11} + x_{12} + x_{13} &= 6 \quad \text{meet the supply of factory at F1} \\ x_{21} + x_{22} + x_{23} &= 9 \quad \text{meet the supply of factory at F2} \end{aligned} \qquad \text{(2 points, 1 point each)}$$

Implicit constraint 1: you ship nonnegative amount of piano

$$x_{11} \geq 0, x_{12} \geq 0, x_{13} \geq 0, x_{21} \geq 0, x_{22} \geq 0, x_{23} \geq 0. \qquad \text{(1 point, 1/6 each)}$$

Implicit constraint 2: you don't make 0.5 piano

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \text{ are integer.} \qquad \text{(1 point, 1/6 each)}$$

So the LP is

$$\begin{aligned}
\min_{x_{11},\dots,x_{23}} \quad & 5x_{11} + 5x_{12} + 3x_{13} + 6x_{21} + 4x_{22} + 1x_{23} \\
\text{s.t.} \quad & x_{11} + x_{21} = 8 \\
& x_{12} + x_{22} = 5 \\
& x_{13} + x_{23} = 2 \\
& x_{11} + x_{12} + x_{13} = 6 \\
& x_{21} + x_{22} + x_{23} = 9 \\
& x_{11} \geq 0, x_{12} \geq 0, x_{13} \geq 0, x_{21} \geq 0, x_{22} \geq 0, x_{23} \geq 0 \\
& x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \text{ are integer.}
\end{aligned} \qquad \text{(1 point for the min)}$$

(10 points if the student directly write down the LP and it is all correct. Otherwise, deduct the points according to the missing portion.)

2 Formulation: knapsack problem (6 points)

You plan to go on a camping trip. There are 5 items you wish to take with you, but together they exceed the 60 weight limit you can carry. You assigned a value to each item:

Item	1	2	3	4	5
Weight	52	23	35	15	7
Value	100	60	70	15	15

Table 2: Weight and value of each item.

Your goal is to determine which item to take to maximize the total value without exceeding the weight restriction. Formulate this problem as a linear program/integer program. State clearly your decision variable(s), objective function and constraint(s).
*You do not need to solve the program.

2.1 Solution

Let $x_1, \dots, x_5$ be the amount of item i to be taken (1 point, 1/5 each). The objective function is

$$100x_1 + 60x_2 + 70x_3 + 15x_4 + 15x_5 \qquad \text{(1 point)}$$

The explicit weight constraint

$$52x_1 + 23x_2 + 35x_3 + 15x_4 + 7x_5 \leq 60 \qquad \text{(1 point)}$$

Implicit constraint 1: cannot pick negative amount

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0 \qquad \text{(1 point, 1/5 each)}$$

Implicit constraint 2: pick integer amount of item

$$x_1, \ldots, x_5 \in \mathbb{N} \qquad \text{(1 point, 1/5 each)}$$

Implicit constraint 3: at most 1 item

$$x_1 \leq 1, x_2 \leq 1, x_3 \leq 1, x_4 \leq 1, x_5 \leq 1 \qquad \text{(1 point, 1/5 each)}$$

Equivalently, the 3 implicit constraints can be written as

$$x_1, \ldots, x_5 \in \{1, 0\}. \qquad \text{(3 points, 3/5 each)}$$

So the LP is

$$\begin{aligned} \max_{x_1,\ldots,x_5} \quad & 100x_1 + 60x_2 + 70x_3 + 15x_4 + 15x_5 \\ \text{s.t.} \quad & 52x_1 + 23x_2 + 35x_3 + 15x_4 + 7x_5 \leq 60 \\ & x_1, x_2, \ldots, x_5 \in \{1, 0\}. \end{aligned} \qquad \text{(1 point for the max)}$$

(6 points if the student directly write down the LP and it is all correct. Otherwise, deduct the points according to the missing portion.)

3 Formulation: employee worksheet (8 points)

You are the boss of a 24-hour supermarket. The supermarket has the following minimal requirements for cashiers:

Period	1	2	3	4	5	6
Time of the day (24-h)	3-7	7-11	11-15	15-19	19-23	23-3
Wages	110	100	100	107	107	110
Minimum no. needed	2	10	4	12	4	2

Table 3: Work period, wages and minimum number of cashiers.

Period 1 follows immediately after period 6. A cashier works 8 consecutive hours, starting at the beginning of one of the six periods. Different periods have different wages. Your goal is to determine a daily employee worksheet which satisfies the requirements with the least wages. Formulate this problem as a linear program/integer program. State clearly your decision variable(s), objective function and constraint(s).
*You do not need to solve the program.

3.1 Solution

Let $x_1, \ldots, x_6$ be the number of cashier beginning work at the start of period i. (1 point, 1/6 each). The objective function is the sum of wages

$$110x_1 + 100x_2 + 100x_3 + 107x_4 + 107x_5 + 110x_6. \qquad \text{(1 point)}$$

There 6 explicit constraints

$$\begin{array}{llllllll} x_1 & & & & & +x_6 & \geq 2 \\ x_1 & +x_2 & & & & & \geq 10 \\ & x_2 & +x_3 & & & & \geq 4 \\ & & x_3 & +x_4 & & & \geq 12 \\ & & & x_4 & +x_5 & & \geq 4 \\ & & & & x_5 & +x_6 & \geq 2 \end{array}$$

(3 points, 3/6 each)

Two implicit constraints: x_i are nonnegative (1 point) and integral (1 point). The LP is

$$\begin{array}{ll} \min_{x_1,\ldots,x_6} & 110x_1 + 100x_2 + 100x_3 + 107x_4 + 107x_5 + 110x_6 \\ \text{s.t.} & x_1 + x_6 \geq 2 \\ & x_1 + x_2 \geq 10 \\ & x_2 + x_3 \geq 4 \\ & x_3 + x_4 \geq 12 \\ & x_4 + x_5 \geq 4 \\ & x_5 + x_6 \geq 2 \\ & x_1, x_2, \ldots, x_6 \text{ are nonnegative and integral} \end{array} .$$

(1 point for the min)

(8 points if the student directly write down the LP and it is all correct. Otherwise, deduct the points according to the missing portion.)

4 Formulation: traveling salesman problem (7 points)

You are a salesman. You need to travel all 5 cities (C1,C2, ..., C5) exactly once to sale your product. The distance between the cities are shown below.

	C1	C2	C3	C4	C5
C1	M	28	57	20	45
C2	28	M	47	46	53
C3	57	47	M	76	85
C4	20	46	M	M	40
C5	45	73	M	40	M

Table 4: Distances between the cities

According to the table, we can let c_{ij} be the distance from city i to city j ($i, j \in \{1, 2, 3, 4, 5\}$). Your goal is to determine the order of visiting that minimize the total distance traveled.

About M Physically the distance between C1 and C1 is zero so naturally we should put $c_{11} = 0$. However this will lead to traveling from C1 to C1 (so we will trap inside C1 forever). Therefore, for proper modeling, M is a big number (e.g., 999).

Traveling Salesman Problem (TSP) is a famous integer programming problem and can be formulated as follows. Let $x_{ij} = 1$ denotes the decision of traveling from city i to city j, and let $x_{ij} = 0$ denotes the decision of NOT traveling from city i to city j. That is, x_{ij} is a nonnegative integer variable in $\{0, 1\}$.

4.1 Using x_{ij} and the table, write down the objective function of this problem (1 point). Is TSP a minimization or maximization problem? (2 points).
Answer: The objective function is the total distance traveled:

$$\sum_{i=1}^{5}\sum_{j=1}^{5} c_{ij}x_{ij}. \qquad \text{(1 point)}$$

It is a minimization problem (1 point).

4.2 A constraint of TSP is that you need to enter all cities exactly once. Write down this constraint mathematically. (1 point).

$$\sum_{i=1}^{5} x_{ij} = 1, \text{ for } j = 1, 2, ..., 5. \qquad \text{(1 point)}$$

because

$$\underbrace{x_{11}}_{\text{C1 to C1}} + \underbrace{x_{21}}_{\text{C2 to C1}} + \underbrace{x_{31}}_{\text{C3 to C1}} + \underbrace{x_{41}}_{\text{C4 to C1}} + \underbrace{x_{51}}_{\text{C5 to C1}} = 1,$$

together with $x_{ij} \in \{0, 1\}$, it means C1 has to be entered at least 1.

4.3 A constraint of TSP is that you need to leave all cities exactly once. Write down this constraint mathematically. (1 point).

$$\sum_{j=1}^{5} x_{ij} = 1, \quad \text{for } i = 1, 2, ..., 5. \qquad \text{(1 point)}$$

Explanation: similar as 2.

4.4 Are the two constraints mentioned above completely describe the constraint set of TSP? Is there any missing constraint? If yes, comment about it (2 point).
No (1 point).
Because the constraints do not eliminate the possibility of forming *subtours*. (1 point)
Example of subtour: $1 \to 2 \to 3 \to 1$ and $4 \to 5 \to 4$, each cities are exactly entered once and leaved once, but this solution is not allowed in TSP since physically you cannot teleport from the first loop into the second loop.
(1 point for "No", and 1 point for subtour, or similar wordings that describe two disjointed subgraphs. If the student can write down the correct expression of subtour, +1 bonus point.)

5 Transform to canonical form and standard form (22 points)

5.1 Transform the LP into canonical form, shows the working steps (8 points).

$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & 0 \\ \text{s.t.} \quad & 1x_1 + 3x_2 = 5 \\ & 2x_1 + 4x_3 \geq 6 \\ & 2 \leq x_2 \leq 4 \end{aligned}$$

Also write down the vector $\mathbf{c}, \mathbf{b}$ and matrix $\mathbf{A}$.
min 0 becomes max 0 (1 point).
$1x_1 + 3x_2 = 5$ becomes $1x_1 + 3x_2 \leq 5$ and $1x_1 + 3x_2 \geq 5 \implies -1x_1 - 3x_2 \leq -5$ (1 point)
$2x_1 + 4x_3 \geq 6$ becomes $-2x_1 - 4x_3 \leq -6$ (1 point)
$2 \leq x_2 \leq 4$ becomes $-1x_2 \leq -2$ and $x_2 \leq 4$ (1 point)
So now

$$\begin{array}{ll} \max\limits_{x_1,x_2,x_3} & [0\ 0\ 0]\mathbf{x} \\ \text{s.t.} & x_1 + 3x_2 \leq 5 \\ & -1x_1 - 3x_2 \leq -5 \\ & -2x_1 - 4x_3 \leq -6 \\ & -1x_2 \leq -2 \\ & x_2 \leq 4 \end{array} \quad \text{or} \quad \begin{array}{ll} \max\limits_{x_1,x_2,x_3} & [0\ 0\ 0]\mathbf{x} \\ \text{s.t.} & x_1 + 3x_2 \leq 5 \\ & -1x_1 - 3x_2 \leq -5 \\ & -1x_1 - 2x_3 \leq -3 \\ & -1x_2 \leq -2 \\ & x_2 \leq 4 \end{array} \qquad \text{(1 point)}$$

and $\mathbf{c} = [0, 0, 0]^\top$ (1 point), and

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ -1 & -3 & 0 \\ -2 & 0 & -4 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -5 \\ -5 \\ -6 \\ -2 \\ 4 \end{bmatrix} \quad \text{or} \quad \mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ -1 & -3 & 0 \\ -1 & 0 & -2 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -5 \\ -5 \\ -3 \\ -2 \\ 4 \end{bmatrix} \qquad \text{(2 points)}$$

5.2 Transform the LP into standard form (7 points)

$$\begin{array}{ll} \min\limits_{x_1,x_2} & 5x_1 + 2x_2 \\ \text{s.t.} & 6x_1 + x_2 \geq 6 \\ & 4x_1 + 3x_2 \leq 12 \end{array}$$

Let $x_1 = x_1^+ - x_1^-$, $x_1^+ \geq 0, x_1^- \geq 0$ and $x_2 = x_2^+ - x_2^-$, $x_2^+ \geq 0, x_2^- \geq 0$, and introduce two slack variables $s_1 \geq 0, s_2 \geq 0$ and the LP becomes

$$\begin{array}{ll} \max\limits_{x_1^+,x_1^-,x_2^+,x_2^-,s_1,s_2} & -5x_1^+ + 5x_1^- - 2x_2^+ + 2x_2^- \\ \text{s.t.} & 6x_1^+ - 6x_1^- + x_2^+ - x_2^- - s_1 = 6 \\ & 4x_1^+ - 4x_1^- + 3x_2^+ - 3x_2^- + s_2 = 12 \\ & x_1^+, x_1^-, x_2^+, x_2^-, s_1, s_2 \geq 0 \end{array}$$

1 point for $x_1 = x_1^+ - x_1^-$, $x_1^+ \geq 0, x_1^- \geq 0$ and $x_2 = x_2^+ - x_2^-$, $x_2^+ \geq 0, x_2^- \geq 0$.
1 point for $s_1 \geq 0, s_2 \geq 0$
1 point for max
1 point for correct objective function.
2 points for equality constraints (1 point each)
1 point for all the inequality constraints

5.3 Transform the LP of 5.1 to standard form (7 points).

The LP of 5.1

$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & 0 \\ \text{s.t.} \quad & 1x_1 + 3x_2 = 5 \\ & 2x_1 + 4x_3 \geq 6 \\ & 2 \leq x_2 \leq 4 \end{aligned} \quad \rightarrow \quad \begin{aligned} \max_{x_1,x_2,x_3} \quad & 0 \\ \text{s.t.} \quad & 1x_1^+ - x_1^- + 3(2 + s_1) = 5 \\ & 2x_1^+ - 2x_1^- + 4x_3^+ - 4x_3^- - s_2 = 6 \\ & s_1 \leq 2 \\ & x_1^+, x_1^-, x_3^+, x_3^-, s_1, s_2 \geq 0 \end{aligned}$$

where 1. $x_1 = x_1^+ - x_1^-$, $x_1^+ \geq 0, x_1^-, \geq 0$.
2. $x_2 = 2 + s_1$, $s_1 \geq 0$
3. $x_3 = x_3^+ - x_3^-$, $x_3^+ \geq 0, x_3^-, \geq 0$.
4. Slack variable s_2 for inequality constraint.
5. min to max
Then, 6. Add slack variable s_3 for $s_1 \leq 2$

$$\begin{aligned} \max_{x_1,x_2,x_3} \quad & 0 \\ \text{s.t.} \quad & 1x_1^+ - x_1^- + 3s_1 = -1 \\ & 2x_1^+ - 2x_1^- + 4x_3^+ - 4x_3^- - s_2 = 6 \\ & s_1 + s_3 = 2 \\ & x_1^+, x_1^-, x_3^+, x_3^-, s_1, s_2, s_3 \geq 0 \end{aligned}$$

1 point for max
3 points for 3 correct equality constraints (1 point each)
1 point for inequity constraints
1 point for appearance of all $x_1^+, x_1^-, x_3^+, x_3^-$
1 point for appearance of all $s_1 s_2, s_3$

6 Solving LP graphically (11 points)

6.1 Draw the constraint set and solve the LP (5 points).

$$\min_{\mathbf{x}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^\top \mathbf{x} \, \text{s.t.} \, \mathbf{x} \in \mathcal{C} = \begin{cases} x_1 + x_2 \geq 1 \\ x_1 + x_2 \leq 2 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$$

4 points for 4 correct edge of the feasible set. 1 point for correct optimal value: =1.

6.2 Make it as simple as you can, write down a LP that has no solution, and draw the constraint set (2 points)

$\max_x x \,\text{s.t.}\, x \geq 1, x \leq -1$ where the feasible set is empty.

6.3 Make it as simple as you can, write down a LP that the constraint set is unbounded, and draw the constraint set (2 points)

$\max_x x \,\text{s.t.}\, x \geq 0.$

6.4 Make it as simple as you can, write down a LP that the optimal point is non-unique, and draw the constraint set (2 points)

$\max_{x_1,x_2} x_1 \text{ s.t. } 0 \le x_1 \le 1, 0 \le x_2 \le 1.$

END of assignment 1.