Solution to CO327 (2021Spring) Assignment 2

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1 Formulation: Fuel purchase policy making (10 points)

Your airline company can buy jet fuel from any one of three vendors. The airline's need for the upcoming month at each of the three airports it serves are 100000 unit at airport 1, 180000 unit at airport 2, and 350000 unit at airport 3. Each vendor can supply fuel to each airport at a price ($ per unit) given by the following table

	Airport 1	Airport 2	Airport 3
Vendor 1	92	89	90
Vendor 2	91	91	95
Vendor 3	78	90	82

Each vendor, is limited in the total number of unit of fuels it can provide during any one month. These capacities are 32000 unit for vendor 1, 270000 units for vendor 2, and 190000 units for vendor 3.

Determine a purchasing policy that will supply the airline's requirement at each airport at minimum total cost, write down the optimization problem (linear program/integer program). State clearly your decision variable(s), objective function and constraint(s).

*You do not need to solve the program

1.1 Solution

3 vendors and 3 airports: 9 decision variable encoding how much each vendor supply fuel to each airport. Let x_{ij} be amount of fuel from vendor i supplied to airport j, the decision variables are

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}. \qquad \text{(1 point, 1/9 each)}$$

The objective function is then the total cost

$$92x_{11} + 89x_{12} + 90x_{13} + 91x_{21} + 91x_{22} + 95x_{23} + 78x_{31} + 90x_{32} + 82x_{33} \qquad \text{(1 point, either 1 or 0)}$$

Explicit constraint 1: meet the demand of airports

$$\begin{aligned} x_{11} + x_{21} + x_{31} &= 100,000 && \text{meet the demand of airport 1} \\ x_{12} + x_{22} + x_{32} &= 180,000 && \text{meet the demand of airport 2} \\ x_{13} + x_{23} + x_{33} &= 350,000 && \text{meet the demand of airport 3} \end{aligned} \qquad \text{(3 points, 1 point each)}$$

Explicit constraint 2: limit capacity of the vendor

$$\begin{aligned} x_{11} + x_{12} + x_{13} &\leq 32000 && \text{meet the supply capacity of vendor 1} \\ x_{21} + x_{22} + x_{23} &\leq 270000 && \text{meet the supply capacity of vendor 2} \\ x_{31} + x_{32} + x_{33} &\leq 190000 && \text{meet the supply capacity of vendor 2} \end{aligned}$$

(3 points, 1 point each. 0.5 point if $\leq$ is changed to $=$)

Implicit constraint 1: nonnegative purchase

$$x_{ij} \geq 0 \ \ \forall i, j \qquad \text{(1 point, 1/9 each)}$$

Implicit constraint 2: integral unit

$$x_{ij} \geq 0 \ \ \forall i, j \text{ are integer.} \qquad \text{(1 point, 1/9 each)}$$

So the LIP is

$$\begin{aligned}
\min_{x_{11},\ldots,x_{33}} \quad & 92x_{11} + 89x_{12} + 90x_{13} + 91x_{21} + 91x_{22} + 95x_{23} + 78x_{31} + 90x_{32} + 82x_{33} \\
\text{s.t.} \quad & x_{11} + x_{21} + x_{31} = 100000 \\
& x_{12} + x_{22} + x_{32} = 180000 \\
& x_{13} + x_{23} + x_{33} = 350000 \\
& x_{11} + x_{12} + x_{13} \leq 32000 \\
& x_{21} + x_{22} + x_{23} \leq 270000 \\
& x_{31} + x_{32} + x_{33} \leq 190000 \\
& x_{ij} \geq 0 \text{ and is integer } \ \forall i, j.
\end{aligned}$$

(1 point for the min)

(10 points if the student directly write down the LIP and it is all correct. Otherwise, deduct the points according to the missing portion.)

2 Formulation: Resource allocation (18 + 2 points)

2.1 Voting visit (5 + 2 points)

You are the next US presidential candidate. You must decide which states to visit in the 10 days before the election. Your goal is to increase the number of votes by the largest possible amount. Your election team provide you the following data

State	Vote ($\times 10^3$) increase by visit	Days required for visit
1	10	4
2	20	3
3	40	3
4	90	4
5	30	3
6	10	1

Formulate this problem. State clearly your decision variable(s), objective function and constraint(s). Bonus (2 points): Which states should be visited? How many votes will be generated by these visits?

2.1.1 Solution

Decision variable: x_i = decision to visit state i or not

$$x_1, \ldots, x_6 \qquad \text{(1 point, 1/6 each)}$$

Objective function: sum of vote increase

$$10x_1 + 20x_2 + 40x_3 + 90x_4 + 30x_5 + 10x_6 \qquad \text{(1 point, either 1 or 0)}$$

Explicit constraint 1: 10 days limit

$$4x_1 + 3x_2 + 3x_3 + 4x_4 + 3x_5 + x_6 \leq 10 \qquad \text{(1 point)}$$

Implicit constraint 1: binary constraint

$$x_i \in \{0,1\} \quad \forall i. \qquad \text{(1 point, 1/6 each)}$$

So the BIP is

$$\begin{aligned} \max_{x_{11},\dots,x_{33}} \quad & 10x_1 + 20x_2 + 40x_3 + 90x_4 + 30x_5 + 10x_6 \\ \text{s.t.} \quad & 4x_1 + 3x_2 + 3x_3 + 4x_4 + 3x_5 + x_6 \leq 10 \\ & x_i \in \{0,1\} \quad \forall i. \end{aligned} \qquad \text{(1 point for the max)}$$

(5 points if the student directly write down the BIP and it is all correct. Otherwise, deduct the points according to the missing portion.)

Bonus: visit states 3,4,5. (1 point) Vote increase: 40+90+30 = 160k. (1 point)

2.2 Voting visit and ads (13 points)

Now you are down to the last 5 days of the campaign. You have $300000 left and three key states appear likely to swing the election one way or other. Each state can be visited, or a TV ad series can be purchased. Your election team provide you the following data

State	Action	Vote ($\times 10^3$) increase by visit	Days required for visit	Cost ($\times 10^3$)
1	Visit	100	4	200
1	Ads	50	0	100
2	Visit	80	4	150
2	Ads	40	0	90
3	Visit	20	1	45
3	Ads	15	0	30

Now assume

- case a: visit and ads on the same state are not mutually exclusive (6 points)
- case b: visit and ads on the same state are mutually exclusive (7 points)

Formulate these problems. State clearly your decision variable(s), objective function and constraint(s). * You do not need to solve the program.

2.2.1 Solution

Case a Decision variable: x_{ij} = decision on state i by action j

$$x_{11}, x_{12}, x_{21}, \dots, x_{32} \qquad \text{(1 point, 1/6 each)}$$

Objective function: sum of vote increase

$$100x_{11} + 50x_{12} + 80x_{21} + 40x_{22} + 20x_{31} + 15x_{32} \qquad \text{(1 point, either 1 or 0)}$$

Explicit constraint 1: 5 days limit

$$4x_{11} + 0x_{12} + 4x_{21} + 0x_{22} + x_{31} + 0x_{32} \leq 5 \qquad \text{(1 point)}$$

Note: those zero term can be omitted.
Explicit constraint 2: 300000 (=300k) budget limit

$$200x_{11} + 100x_{12} + 150x_{21} + 90x_{22} + 45x_{31} + 30x_{32} \leq 300 \qquad \text{(1 point)}$$

Implicit constraint 1: binary constraint

$$x_{ij} \in \{0, 1\} \quad \forall i, j \qquad \text{(1 point, 1/6 each)}$$

So the BIP for case a is

$$\begin{aligned} \max_{x_{11},\dots,x_{33}} \quad & 100x_{11} + 50x_{12} + 80x_{21} + 40x_{22} + 20x_{31} + 15x_{32} \\ \text{s.t.} \quad & 4x_{11} + 0x_{12} + 4x_{21} + 0x_{22} + x_{31} + 0x_{32} \leq 5 \\ & 200x_{11} + 100x_{12} + 150x_{21} + 90x_{22} + 45x_{31} + 30x_{32} \leq 300 \\ & x_{ij} \in \{0, 1\} \quad \forall i, j \end{aligned} \qquad \text{(1 point for the max)}$$

(6 points if the student directly write down the BIP and it is all correct. Otherwise, deduct the points according to the missing portion.)

Case b The BIP for case b is the same as case a with one additional contraint:

$$\textstyle\sum_j x_{ij} \leq 1 \quad \forall i \qquad \text{(1 point, 1/6 each)}$$

(Do not accept = sign.)
The BIP is then

$$\begin{aligned} \max_{x_{11},\dots,x_{33}} \quad & 100x_{11} + 50x_{12} + 80x_{21} + 40x_{22} + 20x_{31} + 15x_{32} \\ \text{s.t.} \quad & 4x_{11} + 0x_{12} + 4x_{21} + 0x_{22} + x_{31} + 0x_{32} \leq 5 \\ & 200x_{11} + 100x_{12} + 150x_{21} + 90x_{22} + 45x_{31} + 30x_{32} \leq 300 \\ & \textstyle\sum_j x_{ij} \leq 1 \quad \forall i \\ & x_{ij} \in \{0, 1\} \quad \forall i, j \end{aligned}$$

(7 points if the student directly write down the BIP and it is all correct. Otherwise, deduct the points according to the missing portion.)

3 Formulation: Tool assignment problem (10+4 points)

You are a miner and you mine two ores called red stone and blue stone. To achieve this, you make two tools: a pickaxe and a drill, out of two material: iron or diamond.

- Because of the weights of the tools to carry, you can only bring one at most pickaxe and one drill for the mining.
- Because of the scarcity of the raw material, you can only make at most one iron tool and at most one diamond tool.

	Made of iron			Made of diamond	
	Pickaxe	Drill		Pickaxe	Drill
Red stone	5	10	Red stone	20	40
Blue stone	10	12	Blue stone	30	50

- Because of the stamina constraint, you can only use a tool to mine one kind of ore. That is, if you choose to use a drill to mine red stone, you cannot use the same drill to mine blue stone.

The table below shows how many ore the tool can mine in the same amount of time.

You task is to assign which tool to make and to use which tool for mining which specific ore, such that the total amount of ore is maximized. Note that, both ores have to be mined. Formulate this as a integer programming problem. State clearly your decision variable(s), objective function and constraint(s).

Bonus (4 points) Solve this problem, state the best option (which type of tool to mine which ore, and what is the total number of ores can be mined), repeat for the **second best** option.

3.1 Solution

8 variables: 2 ores, 2 tools and 2 materials. Let x_{ijk} be mining ore i by tool j made by material k. (convention: $i = 1$ red stone, $i = 2$ blue stone. $j = 1$ pickaxe, $j = 2$ drill, $k = 1$ iron, $k = 2$ diamond) The decision variables are

$$x_{ijk}, \quad i, j, k \in \{1, 2\}. \qquad \text{(1 point, 1/8 each)}$$

The objective function is then the total ore mined

$$5x_{111} + 10x_{211} + 10x_{121} + 12x_{221} + 20x_{112} + 40x_{212} + 30x_{122} + 50x_{222} \qquad \text{(1 point, either 1 or 0)}$$

Explicit constraint 1: you can bring at most one tool per type

$$\begin{aligned} &\textstyle\sum_{ik} x_{i1k} \le 1 \quad \text{bring at most 1 pickaxe} \\ &\textstyle\sum_{ik} x_{i2k} \le 1 \quad \text{bring at most 1 drill} \end{aligned} \qquad \text{(2 points, 1 point each)}$$

Explicit constraint 2: you can make at most one iron tool and at most one diamond tool

$$\begin{aligned} &\textstyle\sum_{ij} x_{ij1} \le 1 \quad \text{make at most 1 iron tool} \\ &\textstyle\sum_{ij} x_{ij2} \le 1 \quad \text{make at most 1 diamond tool} \end{aligned} \qquad \text{(2 points, 1 point each)}$$

Explicit constraint 3: both ore has to be mined

$$\begin{aligned} &\textstyle\sum_{jk} x_{1jk} \ge 0 \quad \text{positive amount of red stone mined} \\ &\textstyle\sum_{jk} x_{2jk} \ge 0 \quad \text{positive amount of blue stone mined} \end{aligned} \qquad \text{(2 points, 1 point each)}$$

Note: $\ge$ replaced by $>$ is ok.
Implicit constraint 1: x_{ijk} is 0-1 integer

$$x_{ijk} \in \{0, 1\} \quad \forall i, j, k \qquad \text{(1 point, 1/8 each)}$$

Note: stating "x_{ijk} is nonnegative" scores nothing if "x_{ijk} is 0-1 integer" is stated. If "x_{ijk} is 0-1 integer" is not stated, then stating "x_{ijk} is nonnegative" scores 1/2 point (1/16 each).

The BIP is

$$
\begin{array}{ll}
\max\limits_{\mathbf{x}} & 5x_{111} + 10x_{211} + 10x_{121} + 12x_{221} + 20x_{112} + 40x_{212} + 30x_{122} + 50x_{222} \\
\text{s.t.} & \sum_{ik} x_{ijk} \leq 1 \ \forall j \\
& \sum_{ij} x_{ijk} \leq 1 \ \forall k \\
& \sum_{jk} x_{ijk} \geq 0 \ \forall i \\
& x_{ijk} \in \{0,1\}
\end{array}
$$

(1 point for the max)

(10 points if the student directly write down the BIP and it is all correct. Otherwise, deduct the points according to the missing portion.)

Bonus part The idea of arriving the solution: treat x_{ijk} as a 2-by-2-by-2 tensor, basically the question is asking to pick which "super-diagonal" of such tenor such that the element-sum is maximized. Solution can be obtained by enumeration (brute-force).

The best option is: use iron pickaxe to mine red stone and use diamond drill to mine blue stone (1 point), total amount of ores = 55. (1 point)

Second best option: use iron pickaxe to mine blue stone and use diamond drill to mine red stone (1 point), total amount of ores = 50 (1 point)

4 Formulation: Set cover problem of Tower defense (29 points)

(This question requires the knowledge of tower defense discussed in lecture 2)

Consider the map in Fig.1 (follows the labeling of the grid). You are allowed to place turret on the labeled blocks, while the red blocks are the path for the enemy. You goal is to determine how to place the least number of turrets, such that each red block is targeted by at least 1 turret.

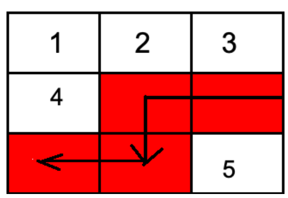

Figure 1: A map.

4.1 Turret can hit all the 4 adjacent blocks (7 points)

Here is the specification of the turret: after placing a turret at a block, the turret can hit all the 4 adjacent blocks – N,E,S,W (North N, East, South, West) simultaneously at once. Formulate the problem as a BIP. Solve this BIP by brute-force and list all the optimal solution(s), if there is no solution, explain why.

Sol

$$
\begin{aligned}
\min_{x_1,\dots,x_5} \quad & \sum_{i=1}^{5} x_i && \text{1 point} \\
\text{s.t.} \quad & x_i \in \{0,1\} \ \forall i && \text{1 point} \\
& x_2 + x_4 \geq 1 && \text{1 point} \\
& x_3 + x_5 \geq 1 && \text{1 point} \\
& x_4 \geq 1 && \text{1 point} \\
& x_5 \geq 1 && \text{1 point}
\end{aligned}
$$

Optimal sol: $x_4 = x_5 = 1, x_1 = x_2 = x_3 = 0$ (1 point)

Potential bonus point In fact x_1 is redundant because there is no constraint on it. If the student is able to remove x_1 from the modeling, or mention $x_1 = 0$ / is redundant, +1 point.

4.2 Rocket launcher (9 points)

Now, beside turrets, you can place rocket launcher in the map. Turrets and rocket launcher are collectively called as "towers". The specifications of the game changed as follows:

- For all the turrets, after placing at a block, the turret can hit all the 4 adjacent blocks (N,E,S,W) simultaneously at once.
- For all the rocket launchers, after placing at a block, the rocket launcher can hit all the 8 adjacent blocks (N,E,S,W,NE,NW,SE,SW) simultaneously at once.
- You can not place a turret and a rocket launcher at the same block.
- Building a turret costs 1 gold, while building a rocket launcher costs 2 gold.

Your goal is to determine how to place the of turrets and rocket launcher with the minimum cost such that each red block is targeted by at least 2 towers. Formulate the problem as a BIP. Solve this BIP by brute-force and list all the optimal solution(s), if there is no solution, explain why.

Sol Let y_i denotes the decision of placing a rocket launcher in grid i. (1 point)

$$
\begin{aligned}
\min_{x_1,\dots,x_5} \quad & \sum_{i=1}^{5} (x_i + 2y_i) && \text{1 point} \\
\text{s.t.} \quad & x_i, y_i \in \{0,1\} \ \forall i && \text{1 point} \\
& x_i + y_i \leq 1 \ \forall i && \text{1 point} \\
& x_2 + x_4 + y_1 + y_2 + y_3 + y_4 + y_5 \geq 2 && \text{1 point} \\
& x_3 + x_5 + y_2 + y_3 + y_5 \geq 2 && \text{1 point} \\
& x_4 + y_4 \geq 2 && \text{1 point} \\
& x_5 + y_4 + y_5 \geq 2 && \text{1 point}
\end{aligned}
$$

There is no sol: the problem is infeasible, because $x_4 + y_4 \leq 1$ and $x_4 + y_4 \geq 2$ contradicts to each other. (1 point)

Potential bonus point In fact x_1 is redundant because there is no constraint on it. If the student is able to remove x_1 from the modeling, or mention $x_1 = 0$ / is redundant, +1 point.

4.3 Another map (13 points)

Consider the map in Fig.2.

(1,1) 1	(1,2) 2	(1,3) 3	(1,4) 2	(1,5) 2
(2,1) 2	(2,2) 2	(2,3) 2	(2,4) 1	(2,5) 1
(3,1) 2	(3,2) 1	(3,3) 3	(3,4) 2	(3,5) 2
(4,1) 2	(4,2) 1	(4,3) 1	(4,4) 1	(4,5) 2
(5,1) 1	(5,2) 2	(5,3) 2	(5,4) 2	(5,5) 2

Figure 2: Another map.

- The map is 5-by-5 grid. Each box is labeled with (i, j) for the coordinate.
- The enemy will walk pass the red blocks, your task is to place turrets on the white blocks to defend the enemy. You cannot place turrets on the red blocks.
- The number in white block is the amount of gold you need to pay if you build one turret on that block. The number in the red block is the number of turrets needed to target such block.
- You can place multiple number of turrets on the same block, and you can place at most 10 turrets in each block.
- After placing a turret at a block, all turrets can hit all the 4 adjacent (N,E,S,W) blocks.

You goal is to determine how to place the turrets with the lowest cost, such that the red path is safe. Write down the optimization problem (linear program/integer program). State clearly your decision variable(s), objective function and constraint(s).
*You do not need to solve the program

Sol Let W denotes the coordinates of the white blocks, and let R denotes the coordinate of the red blocks.

- Decision variables are x_{ij} for $\{i, j\} \in W$, which represents the number of turret placed in the box $\{i, j\}$ (1 point)
- Redundant variables are $x_{11} = x_{15} = x_{25} = 0$. (+1 points)
- Cost function. Let $c_{ij}, \{i, j\} \in W$ be the cost of placing one turrets in the box $\{i, j\}$, where the value c_{ij} follows the map. The cost function is then

$$\sum_{ij \in W} c_{ij} x_{ij}$$

- Constraint 1: $0 \le x_{ij} \le 10$ (2 points)
- Constraint 2: x_{ij} integer (1 point)
- Map constraint
  Elegant way: let $d_{pq}, \{p,q\} \in R$ be the number in the red block
  $$\sum_{(i,j)\,:\,|i-p|\le 1 \text{ and } |j-q|\le 1} x_{ij} \ge d_{pq} \quad \forall (p,q) \in R \quad \text{(8 points + 1 points bonus for short-hand notation)}$$
  OR the **stupid way**:
  $$\begin{array}{lll} (1,3) & x_{12}+x_{14} \ge 3 & (1 \textit{ point}) \\ (2,3) & x_{24}+x_{33} \ge 2 & (1 \textit{ point}) \\ (2,2) & x_{12}+x_{21} \ge 2 & (1 \textit{ point}) \\ (3,2) & x_{31}+x_{33} \ge 1 & (1 \textit{ point}) \\ (4,2) & x_{41}+x_{52} \ge 1 & (1 \textit{ point}) \\ (4,3) & x_{33}+x_{53} \ge 1 & (1 \textit{ point}) \\ (4,4) & x_{34}+x_{54} \ge 1 & (1 \textit{ point}) \\ (4,5) & x_{35}+x_{55} \ge 2 & (1 \textit{ point}) \end{array}$$
- The BIP min (1 point)

5 Computation (8 points)

Transform the following IP

$$\max\ -x_1 + 2x_2 + 3x_3 \ \text{ s.t. } \ 4x_1 - 5x_2 + 6x_3 \ge -7, x_i \text{ is integer}, x_i \ge 0$$

into a integer Knapsack problem in the form

$$\max\ \mathbf{c}^\top \mathbf{x} \ \text{ s.t. } \ \mathbf{A}\mathbf{x} \le \mathbf{b}, \mathbf{x} \text{ is integer vector}$$

where $\mathbf{A}, \mathbf{b}, \mathbf{c}$ are elementwise nonnegative. Write down the $\mathbf{A}, \mathbf{b}$ and $\mathbf{c}$. Solve the Knapsack problem: state the optimal decision variable and the optimal cost value.

Sol Decision variables: replace x_1 by $1 - y_1$

$$y_1, x_2, x_3 \quad \text{or any similar dummy notation} \qquad \text{(1 point)}$$

Cost function

$$\max\ y_1 + 2x_2 + 3x_3 + 2 \quad \text{or} \quad \max\ \mathbf{c}^\top \mathbf{x},\ \mathbf{c} = [1,2,3]^\top \qquad \text{(1 point for c, 1 point for max)}$$

Constraint 1

$$4y_1 + 5x_2 - 6x_3 \le 17 \ \text{ or } \ \mathbf{A} \text{ is a row vector } [4,5,6], \text{ and } b = 17 \qquad \text{(2 points: 1 for A and 1 for b)}$$

Constraint 2

$$y, x_2, x_3 \text{ is integer and } \ge 0 \qquad \text{(2 point)}$$

Optimal solution: $x_3 = +\infty$ (unbounded) (1 point), optimal cost value $= +\infty$ (1 point)
(**The trap: the question only stated that** x_i **is integer and** $x_i \ge 0$, **and there is no upper bound for** x_3, so x_3 can go as large as you want !)

END.