CO327 (2021Spring) Assignment 3

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June 11, 2021

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- Assignment deadline: June-21 23:55.
- Drop your electronic copy (**in pdf, file size ≤4MB, with your name and your student number**) to the dropbox folder in the course page (CO327 - Spring 2021) in WATERLOO LEARN (`learn.uwaterloo.ca`). You will get a penalty if you violate any part of this rule.

1 Quadratic program of MAXCUT (14 points)

Consider the following unndirected graph.

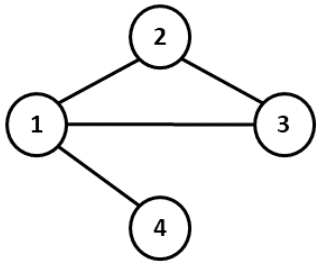

Figure 1: A undirected graph.

A quick review of graph theory: a graph G is a pair of sets V, E, where V is a set whose elements are called vertices, and E is a set of paired vertices, whose elements are called edges. For the graph in Fig.1, $V = \{1, 2, 3, 4\}$ and $E = \{(1,2), (1,3), (1,4), (2,1), (2,3), (3,2)\}$ or simplified as $E = \{(1,2), (1,3), (1,4), (2,3)\}$.

1. Write down the 0-1 adjacency matrix $\mathbf{A}$ of this graph.
2. Consider the MAXCUT problem on this graph, write down the capacity for the problem in the following forms

$$\max_{\mathbf{x}} \underbrace{\frac{1}{2}\sum_{i,j} A_{ij}(1 - x_i x_j)}_{c_1(\mathbf{x})} \text{ s.t. } x_i \in \{-1, +1\}, \quad \text{and} \quad \max_{\mathbf{x}} \underbrace{\frac{1}{4}\sum_{i,j} A_{ij}(x_i - x_j)^2}_{c_2(\mathbf{x})} \text{ s.t. } x_i \in \{-1, +1\}.$$

   that is, write down the cost functions $c_1(\mathbf{x})$ and $c_2(\mathbf{x})$.
3. The capacity can also be expressed as $\mathbf{x}^\top \mathbf{Q} \mathbf{x}$, find $\mathbf{Q}$.
4. Given a solution with max capacity.

2 Linear location planning (19 points)

A country somewhere in Europe is planning on where to build hospital(s). By abstracting the geographical information, the map of the country can be represented by the following weighted undirected graph in Fig.2.

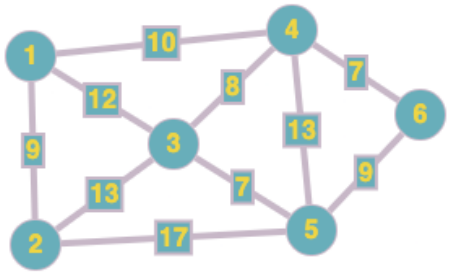

Figure 2: A weighted undirected graph.

For the nodes $i \in V = \{1, 2, \ldots, 6\}$, each node represents a city in this country. The number on the edge $(i, j) \in E$ represents the distance between the two connected cities.

1. Write down E (the set of edges) and the 0-1 adjacency matrix $\mathbf{A}$ of the vertices of this graph.

2. Now, each city can only build at most one hospital. All the hospitals have infinite capacity, and are able to serve (i) all the patients of the city it is located; and (ii) all the patients from its neighbouring cities, if the distance to the neighbouring city is less than or equal to 10.

   Determine the minimum number of hospital to build such that each city is covered by a hospital. Write this down as an optimization problem. *You do not need to solve the problem.

3. Now, the road between city 1 and city 3 has been improved, the number 12 is changed to 2.

   Now, **on top of the conditions stated in Q2,** the country is deciding where to build crematoriums (places that burn dead bodies). Again, each city can only build at most one crematorium. All the crematoriums have infinite capacity, and are able to serve (i) the city it is located; and (ii) all its neighbouring cities (i.e., they are connected by an edge in the graph).

   However, due to religious reasons:

   - People do not want to have a crematorium if a hospital is located there. That is, if there is an hospital located in the city i, there should not be a crematorium.
   - People do not want to have a crematoriums being "too close" to each other. That is, if the **shortest** distance between two cities is less than or equal to 16, then there can be at most 1 crematorium in either one of these two cities.

   Determine the minimum number of hospitals and crematoriums to build such that each city is covered by a hospital and a crematorium. Write this down as an optimization problem. *You do not need to solve the problem.

3 Nonlinear location planning (11 points)

Your oil company wants to build a refinery port that will be supplied from 3 port cities A,B and C:

- Port B is located 300km east and 400km north of Port A,
- Port C is located 400km east and 100km south of Port B,

Your task is to determine the location of the refinery port so that the total amount of pipe required to connect the refinery port to the ports is minimized.

1. Consider a two-dimensional Cartesian coordinate system (x-y plane). Suppose Port A is located at the coordinate (0,0). Draw the locations of Port B and Port C.
2. Suppose the location of the refinery is located at the unknown coordinate (x_1, x_2), write down the total amount of Euclidean distance between the refinery to the ports. (Hint: the Euclidean distance between two points $P(x_1, x_2)$ and $Q(y_1, y_2)$ is $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$).
3. Write down the optimization problem.
4. In this course we do not focus how to solve nonlinear programming problems. Describe how would you approach this problem (explain what you will do on trying to solve this problem).

4 Constrained location planning: Radio placement (40 points)

You are a radio engineer and your task is to determine where to place two large radio towers (with sizes $a_1 \times b_1$ and $a_2 \times b_2$, respectively). Refer to Fig.3, due to some urban constraints, each radio tower can only be placed with in a certain region. For the blue tower, it can not be placed outside the box with the corners $(0, 0), (0, l_{22}), (l_{12}, l_{22})$ and $(l_{12}, 0)$. Similarly, the red tower not be placed outside the box with the corners $(l_{13}, l_{23}), (l_{13}, l_{24}), (l_{14}, l_{24})$ and (l_{14}, l_{23}).

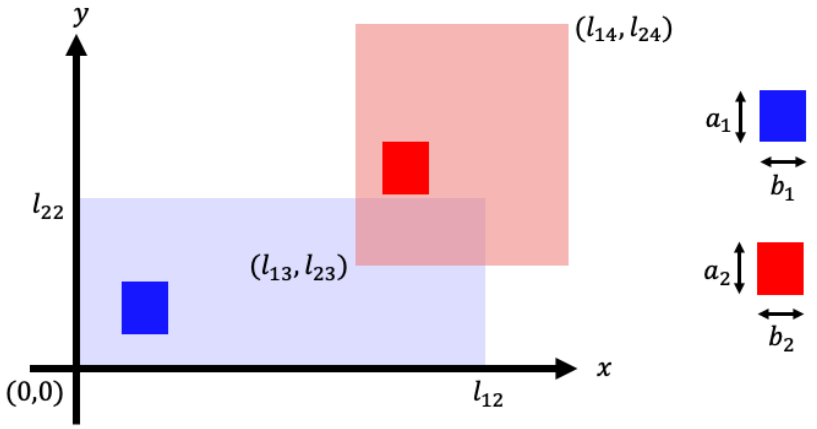

Figure 3: Layout of the feasible region for the radio towers.

To avoid interference of the signal, you want to build the towers as far away as possible. Your job now is to determine where to place these towers such that the "distance" between them is maximized, while making sure that the towers are located within their feasible building ranges.

1. Suppose now we are measuring the distance between the towers using Euclidean distance, write down the optimization problem. (Hint: the Euclidean distance between two points $P(x_1, x_2)$ and $Q(y_1, y_2)$ is $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$).

2. Suppose now we are measuring the distance between the towers using the so-called "min distance", write down the optimization problem. (Hint: the "min distance" between two points $P(x_1, x_2)$ and $Q(y_1, y_2)$ is defined as $\min\{x_1 - y_1, x_2 - y_2\}$).

3. Convert the problem in part 2 into a linear program.

4. Suppose now we are now measuring the distance between the towers using the so-called "taxicab distance", write down the optimization problem. (Hint: the "taxicab distance" between two points $P(x_1, x_2)$ and $Q(y_1, y_2)$ is defined as $|x_1 - y_1| + |x_2 - y_2|$).

5. Convert the problem in part 4 into a linear program. (Hint: you need to introduce new variables.)

5 The bodyguards of president (10 points)

You are the president of the United States. Being such a VIP (Very important person), you are always followed by 3 bodyguards with the following specifications: for each bodyguard,

- For security, the maximum distance between you and the guard cannot exceed 5 meters.
- For convenience, you want to have at least 1 meter of distance between you and the guard.

Now, your task is to arrange the location of the three bodyguards, subject to the specifications listed above. In addition, to prevent forming a "blind spot" (all the guards stay in the same spot), you order the guards to be away from each other as further as possible.

1. Formulate this problem as an optimization problem.
2. Draw the feasible set.
3. Guess what is the geometrical shape of such arrangement, explain why such geometrical shape is optimal.

Note: you only need to consider the problem in a two-dimensional space (x-y plane).

6 Auction market (17 points)

Consider the following betting on horse racing. In the table, blank means 0.

1. Formulate it as the Pari-mutuel model in the original form
2. Formulate the model in 1 to a linear program

Order	Price limit	Quantity limit	Horse 1	Horse 2	Horse 3	Horsre 4	Horse 5
1	0.75	10	1	1	1		
2	0.35	5				1	1
3	0.4	10	1		1		1
4	0.95	10	1	1	1	1	
5	0.75	5		1		1	

7 Formulation of image matching (24 points)

This is a difficult question, so try your best to answer it. Even if you are not sure about the answer, at least write something and do not left it blank.

Introduction In this question, you are going to formulate a quadratic integer program for an engineering problem called "Image Matching" (IM).

In IM, you are given two images, your task is to find the "correspondences" between these two images. A way to find such correspondences is to model the images using graphs. Refer to the example shown in Fig.4, two input images are represented as two graphs respectively. Let $G = (V, E), G' = (V', E')$ be the graphs representing these two images, where the symbol G denotes the graph, the symbol V represents the set of the vertices in the graph, and E represents the set of edges in the graph, and the apostrophe $'$ indicating that we are referring to the second image.

Note: In this question, you do not need to worry about how G and G' are obtained, you assume someone working in a tech company have already prepared G and G' for you, i.e., you are given G and G' in the beginning. Furthermore, you are given a set of values s_V and s_E, to be discussed later.

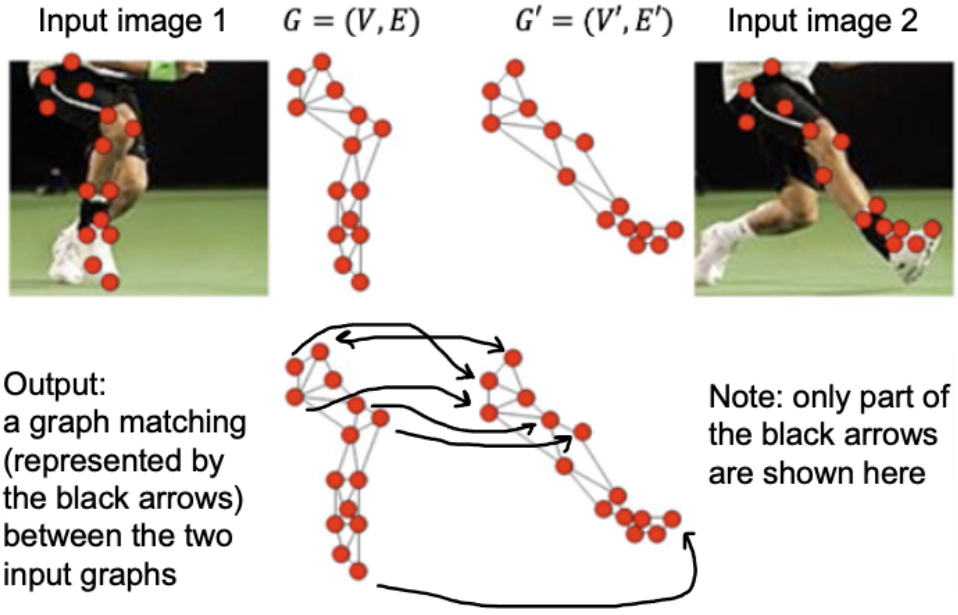

Figure 4: An illustrative example of image matching. Top row: two input images and their graph representation. Bottom row: the output of solving a QIP, which is the matching between the vertices of the two graphs. The image is modified from a research paper.

On the decision variable Given G and G', your job is to determine the correspondence between V and V'. That is, for the n nodes in V and n' nodes in V', your task is to determine which node in V' is connected to the nodes in V.

Q1. Based on the above description, what is the decision variable for this problem?

The goal of IM is to determine the correspondence between V and V'. One way to determine such correspondence is to model it by forming an optimization problem on maximizing a cost function subject to some constraints.

On the cost function To model the cost function for IM, a common approach is to use a "similarity score" that measures the similarity between the two graphs. As a graph has two parts (the vertices and the edge), so the similarity score has two parts:

- Thee first part represents the similarity between two vertices: one vertex from V and one vertex from V'. Mathematically, for a vertex-pair (v_i, v_a), where $v_i \in V$ and $v_a \in V'$, the score function can be represented as c_{ia}.
- The second part represents the similarity between two edges: one edge from E and one edge from E'. Mathematically, for an edge-pair (e_{ij}, e_{ab}), where $e_{ij} \in E$ and $e_{ab} \in E'$, the score function can be represented as $d_{ia,jb}$.

You do not need to worry about how the values in c and d are obtained, you assume someone working in a tech company have already prepared the values for you.

2. Based on the above description, write down the cost function.

On the constraint Now we talk about the constraint. There is only a simple constraint in IM, the matching has to be "one-to-one", that is, for all v in V, it has to be matched to exactly one v' in V', and for all v' in V', it has to be matched to exactly one v in V

3. Based on the above description, write down the constraints.

The optimization problem **4. Write down the whole optimization problem.**

On matrix form representation of cost function The similarity functions are are usually represented by a symmetric matrix $\mathbf{A}$, where a non-diagonal element $A_{ia,jb} = d_{ia,jb}$ contains the edge similarity of two correspondences (v_i, v'_a) and (v_j, v'_b), and a diagonal term $\mathbf{A}_{ia,ia} = c_{ia}$ represents the node similarity of a correspondence (v_i, v_a).

5. Based on the above description, rewrite down the cost function and write down the optimization problem again with the new cost function.

8 Administrative stuffs (0 points)

Follows the instructions below.

1. Create an account in Overleaf `https://www.overleaf.com/`.
2. Create an account in MATLAB `https://www.mathworks.com/mwaccount/register` using your UWaterloo email.

END of assignment.